MTH 320: Presentation problems

Chapter 1:

~~1.4.1~~ (*)[1] Prove that there is a real number α satisfying $\alpha^4 = 2$, using proof ideas from Theorem 1.4.5. There is a hint towards the end of that proof. *(Presented by AC, Monday, March 18.)*

1.4.2 (**) Prove that the set $\{\frac{k}{2^n} : k, n \in \mathbb{N}\}$ is dense in $\mathbb{R}$.

~~1.4.3~~ (***) Show that the collection of all finite subsets of $\mathbb{N}$ is a countable set. *(Presented by AS, Friday, February 1)*

Chapter 2:

~~2.3.1~~ (*) Does Theorem 2.3.4(i) remain true if all of the inequalities are assume to be strict? If so, prove it. If not, find two counter examples. *(Presented by RV, Friday, February 8)*

~~2.3.2~~ (**) Show that if (x_n) is a convergent sequence, then the sequence (y_n) defined by taking the averages

$$y_n = \frac{x_1 + \cdots + x_n}{n}$$

also converges to the same limit. Using the sequence of partial sums $x_n = \sum_{i=0}^{n}(-1)^n$, show that the sequence of averages converges in certain cases even when (x_n) does not. *This provides a generalization of our classical notion of convergence.*

~~2.3.3~~ (*) Consider the doubly indexed array $a_{m,n} = \dfrac{m}{m+n}$.

(a) What should $\lim\limits_{m,n\to\infty} a_{m,n}$ represent? Compute (no proof necessary) the iterated limits

$$\lim_{m\to\infty}\lim_{n\to\infty} a_{m,n} \text{ and } \lim_{n\to\infty}\lim_{m\to\infty} a_{m,n}.$$

(b) Formulate a rigorous definition for the statement $\lim\limits_{m,n\to\infty} a_{m,n} = L$.

~~2.4.1~~ (**) Let $x_1 = 1$, $x_2 = \sqrt{2}$, $x_2 = \sqrt{2\sqrt{3}}$, and in general let

$$x_n = \sqrt{2\sqrt{3\sqrt{4\cdots\sqrt{n}}}}.$$

If (x_n) converges, prove it. If you proved this, do you have a guess for the limit? If the sequences diverges, prove it. *(Presented by AB.)*

[1] If you make some mistakes with ** or *** difficulty problems, it won't be counted against you as much as with * difficulty problems.

2.5.1 (** but requires good understanding of sequences) Let (x_n) be a sequence of real numbers. Prove that $x_n \to L$ if and only if for all subsequences (x_{n_i}) of (x_n), there is a subsequence $(x_{n_{i_k}})$ of (x_{n_i}) that converges to L.

~~2.6.1~~ (*) Assume (a_n) and (b_n) are Cauchy sequences. Prove $c_n = |a_n - b_n|$ is a Cauchy sequence. *(Presented by AG, Friday, February 8)*

2.7.1 (**) (Dirichlet's test) If the partial sums of $\sum_{n=1}^{\infty} x_n$ are bounded (but not necessarily convergent... think of an example?), and if (y_n) is a sequence satisfying $y_1 \geq y_2 \geq \cdots \geq 0$ with $\lim y_n = 0$, then the series $\sum_{n=1}^{\infty} x_n y_n$ converges. Proof sketch:

(a) Let $M > 0$ be an upper bound for the partial sums of $\sum_{n=1}^{\infty} x_n$. Use summation by parts (Hw5) to show that

$$\left| \sum_{j=m+1}^{n} x_j y_j \right| \leq 2M|y_{m+1}|.$$

(b) Prove Dirichlet's Test just stated.

(c) Show how the alternating series test (Hw5) can be derived as a special case of Dirichlet's test.

2.7.2 (**) (Abel's test) If $\sum_{n=1}^{\infty} x_n$ converges, and if (y_n) is a sequence satisfying $y_1 \geq y_2 \geq \cdots \geq 0$, then $\sum_{n=1}^{\infty} x_n y_n$ converges. A proof sketch:

(a) Assume that $\sum_{n=1}^{\infty} a_n$ has partial sums that are bounded by a constant $A > 0$, and assume $b_1 \geq b_2 \geq \cdots \geq 0$. Use summation by parts (Hw5) to show

$$\left| \sum_{j=1}^{n} a_j b_j \right| \leq 2Ab_1.$$

(b) For fixed $m \in \mathbb{N}$, apply part (a) to $\sum_{j=m+1}^{n} x_j y_j$ by setting $a_n = x_{m+n}$ and $b_n = y_{m+n}$. Argue that an upper bound on the partial sums of $\sum_{n=1}^{\infty} a_n$ can be made arbitrarily small by taking large m.

2.8.1 (***) (Fubini-Tonelli Theorem) Let $\{a_{ij} : i, j \in \mathbb{N}\}$ be a doubly indexed array of real numbers. If

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|$$

converges, then $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$ converges and moreover

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij} = \lim_{n \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}.$$

2.8.2 (**) Give an example of a doubly indexed array $\{a_{ij} : i, j \in \mathbb{N}\}$ such that no two of the following are equal:

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}, \qquad \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}, \quad \text{and} \lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}.$$

As a bonus (not necessary for the presentation) find an array so that all three of the above limits are finite. As another bonus, can you make one equal to infinity, another negative infinity, and the third zero? Is it possible to make all three different but the same sign (positive, negative, or zero)?

Chapter 3:

~~3.1.1~~ (***) Construct a Cantor set that does not have zero length (but still nowhere dense). What is its dimension, using a heuristic like the table on page 78? *(Presented by NT on 2/20.)*

3.2.1 (***)

(a) It is a fact that in $\mathbb{R}$, the only closed and open sets are $\varnothing$ and $\mathbb{R}$. What are some sets in $\mathbb{R}$ that are both F_σ and G_δ?

(b) Is the set $\mathbb{Q}$ a G_δ set? Is it F_σ? What about the irrational numbers, $\mathbb{R} \setminus \mathbb{Q}$? Prove all four of the claims.

(c) Can you construct a set that is not closed, open, F_σ, nor G_δ? (Or must all subsets of $\mathbb{R}$ be either open, closed, F_σ or G_δ?)

~~3.2.2~~ (*or ***) Prove or find a counterexample: $\mathbb{R}$ is the only open set containing $\mathbb{Q}$. *(Presented by IM on 2/20.)*

~~3.3.1~~ (**) Show the Cantor set is compact. *(Presented by BW.)*

3.3.2 (*) For two subsets of real numbers A, B define

$$A + B = \{a + b : a \in A, b \in B\}.$$

Show that $C + C = [0, 2]$, where C is the Cantor set. (Proof outline provided in exercises of ch.3 section 3.)

3.3.3 (**) Call a set "clompact" if every closed cover has a finite subcover. Characterize the clompact subsets of $\mathbb{R}$.

3.4.1 (*) Let $\mathbb{Q} = \{r_1, r_2, \dots\}$ be an enumeration of the rational numbers. Let $\varepsilon_n = 1/2^n$. Define

$$O = \bigcup_{n=1}^{\infty} B_{\varepsilon_n}(r_n).$$

Show that the complement of O is closed, nonempty, and consists of only irrational numbers.

3.4.2 (**) If A is connected, does that imply $\bar{A}$ is connected? If A is perfect, does that imply $\bar{A}$ is perfect?

3.5.1 (**) If A is F_σ, does that imply $\bar{A}$ is F_σ? If A is G_δ, does that imply $\bar{A}$ is G_δ?

Chapter 4:

4.1.1 (*) Let A be a set consisting only of isolated points. Construct a function whose set of discontinuities is A.

4.2.1 (*) Show that $\lim_{x\to c} f(x) = L$ iff for all $\varepsilon > 0$ there is $\delta > 0$ such that

$$|x - c| \leq \delta \implies |f(x) - L| \leq \varepsilon.$$

4.2.2 (**) Let $f : \mathbb{R} \to \mathbb{R}$. Define the closed set

$$D_\varepsilon(x) = \{y \in \mathbb{R} : |x - y| \leq \varepsilon\}.$$

Prove f is continuous at x iff the inverse image of D_ε is closed.

4.3.1 (**) If f, g, h are continuous on the real line, prove that $f(g(h(x)))$ is continuous using the definition of continuity.

4.3.2 (*) Consider

$$f(x) = x^a \sin(1/x^b).$$

For what a, b is $f(x)$ continuous (everywhere)?

~~4.3.2~~ (**) Let C be the cantor set. Define the function $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & x \in C, \\ 0 & x \notin C. \end{cases}$$

Prove f is discontinuous at all $x \in C$ and is continuous at all $x \notin C$. *(Presented by YW.)*

~~4.4.1~~ (*) Assume f is continuous and O is an open set. Is $f(O)$ open? If K is a closed set, is $f(K)$ closed? *(Presented by GA.)*

4.4.2 (**) In general, is $f(A \cup B) = f(A) \cup f(B)$? Is $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$? Here, $f^{-1}(A)$ denotes the inverse image of A. If either of the above are false, modify them to make them true (add additional hypotheses or change $=$ to $\subseteq, \supseteq$) .

4.4.3 (**) In general, is $f(A \cap B) = f(A) \cap f(B)$? Is $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$? Here, $f^{-1}(A)$ denotes the inverse image of A. If either of the above are false, modify them to make them true (add additional hypotheses or change $=$ to $\subseteq, \supseteq$).

4.5.1 (**) If A is perfect and f is continuous, is $f(A)$ perfect?

~~4.5.2~~ (*) If A is disconnected, and f is continuous, is $f(A)$ disconnected? *(To be presented by SG.)*

~~4.5.3~~ (*) If A is finite/countable/uncountable, is $f(A)$ finite/countable/uncountable?

4.5.4 (**) If A is F_σ and f is continuous, is $f(A)$ F_σ?

4.5.5 (**) If A is G_δ and f is continuous, is $f(A)$ G_δ?

4.5.6 (***) Let $f : [a, b] \to \mathbb{R}$ We say f has the Intermediate Value Property if for all $x < y$ in $[a, b]$ and all L between $f(x)$ and $f(y)$, there is $x < c < y$ such that $f(c) = L$. Prove that if f has the IVP and is monotone, then f must be continuous.

~~4.5.7~~ (**) Come up with a function different from the book's example that satisfies the Intermediate Value Property but is not continuous. Note that your example cannot be monotone, otherwise the IVP would imply continuity. (Presented by JH.)

4.6.1 (***) Let F be an F_σ set. Modify the construction of Thomae's function to come up with a function whose set of discontinuities is F.

Chapter 5:

~~5.3.1~~ (**) Prove $|\cos(x) - \cos(y)| \leq |x - y|$. *(Presented by AG.)*

5.3.2 (**) Assume $f : (-1, 1) \to \mathbb{R}$ is twice differentiable. Prove that for all x there is ξ between 0 and x such that

$$f(x) = f(0) + f'(0)x + f''(\xi)x^2/2.$$

5.4.1 (***) Prove the nowhere differentiable function we constructed is not differentiable at points of the form $p/2^k$.

5.4.2 (***) Does the construction in this section still work for

$$\sum_{n=0}^{\infty} 2^{-n} f(3^n x)?$$

How about

$$\sum_{n=0}^{\infty} 3^{-n} f(2^n x)?$$

Chapter 6:

6.2.1 (**) Let

$$g_n(x) = \frac{nx + \sin(nx)}{2n}.$$

Find the pointwise limit of (g_n) on $\mathbb{R}$. Is the convergence uniform on $[-1, 1]$? Is the convergence uniform on $\mathbb{R}$?

6.2.2 (**) Construct a sequence of continuous functions (f_n) on $[-1, 1]$ that converges to a limit function that is not bounded on this set.

6.3.1 (**) Let

$$f_n(x) = \frac{nx^2 + 1}{2n + x}.$$

Find $f(x) = \lim f_n(x)$, and then take the derivative of f. Next, compute f_n' and show this sequence converges uniformly on $[-10, 10]$. Use the theorem from this chapter to conclude $f' = \lim f_n'$.

6.4.1 (***) Let $\mathbb{Q} = \{r_1, r_2, \ldots\}$ be an enumeration of the rational numbers. For each $n \in \mathbb{N}$, define

$$u_n(x) = \begin{cases} 1/2^n & \text{if } x > r_n \\ 0 \text{ if } x \leq r_n. \end{cases}$$

Now, let $h(x) = \sum_{n=1}^{\infty} u_n(x)$. Prove h is monotone, defined on all of $\mathbb{R}$, and is continuous at every irrational point.