MTH320 Final exam review questions, Spring 2019

Chapter 1:

(a) Use induction to prove

$$(A_1 \cap \cdots \cap A_n)^c = A_1^c \cup \cdots \cup A_n^c.$$

Why can't induction be used to prove

$$\left(\bigcap_{n=1}^{\infty} A_n \right)^c = \bigcup_{n=1}^{\infty} A_n^c?$$

(b) If $g : \mathbb{R} \to \mathbb{R}$, prove that $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ and $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$.

(c) Is $\mathbb{Q}$ countable or uncountable? Provide an argument either way.

Chapter 2:

(a) If $b_n \to b$, prove $|b_n| \to |b|$. If $|b_n| \to |b|$, is it true that $b_n \to b$?

(b) Use the definition of limit to compute

$$\lim_{n \to \infty} \frac{3n + 1}{8n - 4}.$$

(c) If $\sum_{n=1}^{\infty} a_n$ converges, and $\sum_{n=1}^{\infty} b_n$ converges, does $\sum_{n=1}^{\infty} a_n b_n$ converge? Provide a proof if it's true, or a counterexample if it's not true.

Chapter 3:

(a) We say $A \subset \mathbb{R}$ is dense in $B \subset \mathbb{R}$ if $\overline{A} \cap B = B$. Prove that if A is dense in B, and B is dense in $\mathbb{R}$, then A is dense in $\mathbb{R}$.

(b) Let

$$B = \left\{ \frac{n(-1)^n}{n+1} : n \in \mathbb{N} \right\}.$$

Find the limit points of B. Is B closed? Is B open?

(c) Provide an example of a set $A \subset \mathbb{R}$ that is neither open nor closed. Prove that it is neither open nor closed.

Chapter 4:

(a) Prove or disprove: $\lim_{x \to c} f(x) = L$, implies $f(c) = L$.

(b) If f is continuous at c but g is not continuous at c, prove $f + g$ is not continuous at c. If f and g are both discontinuous at c, is $f + g$ discontinuous at c?

(c) Prove that $\sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Chapter 5:

(a) Let $f : (a, b) \to \mathbb{R}$ be differentiable. Prove that if $f'(x)$ is not a constant function, then $f'(c)$ is irrational for some $c \in (a, b)$.

(b) State the Mean Value Theorem. Use it to prove that if f is differentiable on $A \subset \mathbb{R}$, and if $f'(x)$ is bounded by M for all $x \in A$, then $f(x)$ is uniformly continuous on A.

(c) Provide an example of a continuous function on A that is differentiable on A, but not twice differentiable on A.

Chapter 6:

(a) If $f_n(x)$ converges uniformly to $f(x)$ on (a, b), and if $f_n(a) \to f(a)$, $f_n(b) \to f(b)$, does f_n converge uniformly to f on $[a, b]$?

(b) Let $f_n(x) = 1/(x^2 + n^2)$. Prove $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $\mathbb{R}$.

(c) Find the 2nd order Taylor approximation of $\sin(x)$ on $[-1, 1]$. What is the bound on the error term $E_2(x)$? What about $E_3(x)$?