1. (5 points each) Define and provide an example. Choose 4 out of 5 of the following:

(i) Limit point.

(ii) Isolated point.

(iii) Open set.

(iv) Closed set.

(v) $f : \mathbb{R} \to \mathbb{R}$ continuous at x.

Solutions:

(i) We say x is a limit point of a set $A \subset \mathbb{R}$ if for all $r > 0$, the ball of radius r around x contains some other point of A. In other words,

$$B_r(x) \setminus \{x\} \cap A \neq \varnothing.$$

Example: 0 is a limit point of $(0, 1)$. Note that .5 and everything in $(0, 1)$ are also limit points of $(0, 1)$.

(ii) A point $x \in A$ is called an isolated point of A if there is some $r > 0$ such that

$$B_r(x) \cap A = \{x\}.$$

Note the subtlety: something is an isolated point of A only if it is in A, unlike a limit point.

Example: $A = \{4\}$. 4 is an isolated point of A.

(iii) $A \subset \mathbb{R}$ is open if for all $x \in A$ there is some $r > 0$ such $B_r(x) \subseteq A$.

Examples: $(0, 1), \varnothing, \mathbb{R}, (0, \infty)$.

(iv) A is closed if its complement is open. Or, A is closed if it contains all its limit points.

Examples: $[0, 1], \varnothing, \mathbb{R}, [0, \infty)$.

(v) f is continuous at c if

$$\lim_{x \to c} f(x) = f(c).$$

Or, f is continuous at c if for all $\varepsilon > 0$, there is $\delta > 0$ such that

$$|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon.$$

Example: $f(x) = x$.

2. (5 points each) Prove or disprove:

(i) If f is continuous on $\mathbb{R}$, then $f\big((a,b)\big)$ is an interval.

(ii) The Cantor set is open.

(iii) Every function defined on the integers $\mathbb{Z}$ is continuous.

(iv) If f is uniformly continuous and g is uniformly continuous, then their product is uniformly continuous.

Solutions:

(i) We had a theorem that said if f is continuous, then

$$f(\text{conected set}) = \text{connected set}.$$

We had another theorem that said the connected sets in $\mathbb{R}$ are just intervals. Since (a,b) is connected, its image under the continuous function f must also be connected, and therefore an interval.

(ii) The Cantor set $C \subset [0,1]$ is not open because any ball around 1 contains points bigger than 1. Therefore no ball around 1 is completely contained in C. So C is not open since $1 \in C$ (by the construction, 1 appears at every step C_n, or, recall that any end point at any step C_n is contained in C).

(iii) This was from the homework. Let $\varepsilon > 0$. Let $\delta = .5$. Then if $x, y \in \mathbb{Z}$,

$$|x-y| < \delta \implies x = y.$$

Therefore

$$|f(x) - f(y)| = 0 < \varepsilon,$$

so f is continuous by definition.

(iv) False. Counterexample: $f(x) = x = g(x)$. See the next question for the proof that x^2 is not uniformly continuous.

Note that $f(x) = x$ is continuous because for all ε, we can let $\delta = \varepsilon$. Then,

$$|x-y| < \delta = \varepsilon \implies |f(x) - f(y)| = |x-y| < \varepsilon.$$

3. (10 points each)

(a) Prove $f(x) = x^2$ is continuous everywhere using the definition.

(b) Prove f is not uniformly continuous on $\mathbb{R}$.

(c) Is f uniformly continuous on $(0, 100)$? (Prove or disprove.)

Solutions:

(a) If $x = 0$, we can choose $\delta = \varepsilon^{1/2}$.

Otherwise, let $\delta = \min\{|x|, \varepsilon/(3|x|)\} > 0$. Then,

$$|x^2 - y^2| = |x - y||x + y| < \frac{\varepsilon|x + y|}{3|x|}.$$

By triangle inequality,

$$|x + y| \leq |x| + |y|$$

and

$$|y| < |y - x| + |x| < \delta + |x|.$$

Therefore, the above is less than or equal to

$$\varepsilon\frac{|x| + |y|}{3|x|} \leq \varepsilon\frac{|x| + \delta + |x|}{3|x|} \leq \varepsilon.$$

The last inequality follows from $\delta = \min\{|x|, *\} \leq |x|$.

(b) Let $x_n = n$ and let $y_n = n + 1/n$. Then $|x_n - y_n| = 1/n \to 0$. However,

$$|f(x_n) - f(y_n)| = |n^2 - (n + 1/n)^2| = 2 + 1/n^2 \to 2 \neq 0.$$

By a theorem from the book, this shows f is not uniformly continuous.

(c) Since f is continuous, it is uniformly continuous on any compact set. In particular, f is uniformly continuous on $[0, 100]$.

Therefore, for all $x, y \in [0, 100]$, for all $\varepsilon > 0$, there is $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

Restricting our attention to only those $x, y \in (0, 100)$ proves that for all $\varepsilon > 0$, the same δ as above works.

4. (10 points each)
We say $f : A \to \mathbb{R}$ is Lipschitz if there exists $M \geq 0$ such that for all $x, y \in A$,

$$|f(x) - f(y)| \leq M|x - y|.$$

(a) Show that if f is Lipschitz, then f is uniformly continuous.

(b) Recall that a is a fixed point of f if $f(a) = a$. Provide an example of a Lipschitz function defined on $\mathbb{R}$ that does not have a fixed point. (Contrast with contraction mappings, where $M < 1$.)

(c) Provide an example of a uniformly continuous function that is not Lipschitz. (Hint: enough to show not Lipschitz at a single point y. Think about why a function would fail to be Lipschitz.)

Solutions:

(a) If $M = 0$, then $|f(x) - f(y)| \leq 0$. Therefore, for all ε, you can choose absolutely any $\delta > 0$ so that

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

Otherwise, if $M > 0$, let $\delta = \varepsilon/M$. Then

$$|f(x) - f(y)| \leq M|x - y| < M\delta = \varepsilon.$$

(b) Let $f(x) = x + 1$. Clearly $f(x) \neq x$ for any x.

(c) Let $f : [0, 1] \to [0, 1]$ be defined by $f(x) = \sqrt{x}$. This is an example of a uniformly continuous function that is not Lipschitz:

- f is uniformly continuous since it is continuous on a compact set.
- letting x be arbitrary but $y = 0$, we show that there can be no such M, satisfying the following, and therefore f cannot be Lipschitz:

$$\sqrt{x} = |\sqrt{x} - \sqrt{0}| = |f(x) - f(y)| \leq M|x - y| = M|x| = Mx.$$

There is no fixed number M such that $\sqrt{x}/x = 1/\sqrt{x} \leq M$ for all $0 < x < 1$.