1. Provide examples of the following:

(i) (4 points) Convergent sequences $(a_n), (b_n)$ such that (a_n/b_n) does not converge to a real number.

(ii) (4 points) Divergent sequences $(a_n), (b_n)$ such that $(a_n + b_n)$ converges.

(iii) (2 points) True or false: if $\sum_{n=1}^{\infty} a_n$ converges absolutely and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n b_n$ converges.

(Extra credit, 5 points: prove your claim.)

(Extra credit, 5 points: come up with a counterexample if $\sum a_n$ converged conditionally).

(iv) (2 points) True or false: if $\sum_{n=1}^{\infty} a_n$ diverges and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n + b_n$ diverges.

Solutions:

(i) Recall we had an exercise to show that (a_n/b_n) converges if a_n converges and b_n converges to a nonzero real number. Therefore, to provide an example of (a_n/b_n) diverging, we need $b_n \to 0$. For example, let $b_n = 1/n$ and let $a_n = 1$ for all n (constant sequences are the simplest ones – the only example that would not work here is $a_n = 0$).

So $a_n/b_n = 1/(1/n) = n$. The sequence (n) diverges does not converge to any real number.

(ii) Remember when we used cancellation of multiplication to construct $(a_n b_n)$ that converged for two divergent sequences $(a_n), (b_n)$. We can try to cancel in this example too: for any sequence (a_n), we can let $b_n = -a_n$, so that $a_n + b_n = 0$.

More explicitly, let $a_n = n$ and let $b_n = -n$. Then $(a_n), (b_n)$ both diverge but $(a_n + b_n)$ is the constant sequence (0), which converges.

(iii) This is true. We use the Cauchy criterion for sums to prove this.

Let $\varepsilon > 0$.

Since $\sum a_n$ converges absolutely, there is N_1 such that $n > m > N_1$ implies

$$\sum_{j=m+1}^{n} |a_j| < \varepsilon.$$

Since $\sum b_n$ converges, we know that $b_n \to 0$. So let N_2 be such that $n \geq N_2$ implies $|b_n| < 1$.

Let $\varepsilon > 0$ and let $N = \max\{N_1, N_2\}$. Then if $n > m > N$, we have

$$\left| \sum_{j=m+1}^{n} a_j b_j \right| \leq \sum_{m+1}^{n} |a_j b_j| < \sum_{j=m+1}^{n} |a_j| < \varepsilon.$$

The first inequality is the triangle inequality applied $(n - (m+1))$ times. The second inequality is because for each $m + 1 \leq j \leq n$,

$$|a_j b_j| = |a_j||b_j| < |a_j| \cdot 1$$

since $m + 1 > N \geq N_2$. The last inequality is because $n > m > N \geq N_1$.

A counterexample could be $a_n = b_n = (-1)^{n+1}/\sqrt{n}$. Both sums converge but their product is $1/n$, whose sum diverges.

(iv) False, take $a_n = 1$, $b_n = -1$.

Given a sequence (a_n), recall the definition of $\limsup a_n$ and $\liminf a_n$:

$$\limsup a_n = \lim_{n\to\infty}(\sup\{a_k : k \geq n\}).$$

$$\liminf a_n = \lim_{n\to\infty}(\inf\{a_k : k \geq n\}).$$

2. Provide examples of the following:

(i) (4 points) A sequence (a_n) with $\liminf a_n = 1$ and $\limsup a_n = 2$.

(ii) (4 points) A sequence (a_n) such that $\limsup a_n = \infty$ and $\liminf a_n = -\infty$.

(iii) (3 points) A sequence (a_n) such that $\limsup a_n = -\infty$.

(iv) (2 points) True or false: if (a_n) converges, then $\limsup a_n = \liminf a_n$.

Extra credit:

- (1 point) Find a subset S of real numbers such that $\sup S = -\infty$ and $\inf S = \infty$. (Hint: If S any set containing the number 2019, can this be hold?)
- (1 point) For the set S you found, does it make sense to think of it as a sequence ask what is $\limsup S$?
- (2 point) For subsets $S \subset \mathbb{R}$, $\inf S \leq \sup S$ if and only if what?

Solutions:

(i) $(1, 1, 2, 2, 1, 1, 2, 2, ...)$.

(ii) $(1, -1, 2, -2, 3, -3, 4, -4, ...)$.

(iii) $(-1, -2, -3, -4, ...)$.

(iv) True. This was a homework problem.

Extra credit:

- The empty set $\varnothing$ satisfies $\sup\varnothing = -\infty$ and $\inf\varnothing = \infty$. This is weird, but it's true. Clearly for all elements x of the empty set (there are none) $x < 10$ is trivially satisfied. So 10 is an upper bound. This works for all numbers, and there will never be a smallest real number for which this argument fails, so the least upper bound must be $-\infty$. The same argument works for $\inf\varnothing = \infty$.
- (1 point) There is no such thing as an empty sequence, because by definition, a sequence has to be a countably infinite list.
- (2 point) $\inf S \leq \sup S$ iff S is nonempty. In many places in math, the empty set is often an exception, just like with multiplication, division, exponentiation, zero is often an exception to commonly stated rules. (E.g. anything to the zero is one, but zero to anything is zero, so what is 0^0?)

3. (10 points each, total 20) Prove or provide a counterexample for two of the three statements below.

(a) If (a_n) converges to zero and (b_n) is bounded, $(a_n b_n)$ converges.

(b) If (a_n) converges, (b_n) is bounded, then $(a_n b_n)$ converges.

(c) If (a_n) converges, (b_n) is bounded and decreasing, $(a_n b_n)$ converges.

Solutions:

(a) True. Let $\varepsilon > 0$. Since b_n is bounded, there is a real number $M > 0$ such that $|b_n| \leq M$. Since a_n converges to 0, let N be such that $n \geq N$ implies $|a_n - 0| < \varepsilon/M$. Then, for $n \geq N$ we have

$$|a_n b_n - 0| = |a_n b_n| < M|a_n| < M\varepsilon/M = \varepsilon.$$

This proof also shows that $a_n b_n \to 0$. You could have also used the Cauchy criterion for a_n and boundedness of b_n to show convergence without finding the explicit limit.

(b) This turns out to be false. For example, let $a_n = (1, 1, 1, 1, ...)$, which obviously converges, and let $b_n = (1, 0, 1, 0, 1, 0, ...)$, which is clearly bounded but does not converge. Then $a_n b_n = b_n$.

(c) Here we can use a result from class. Since b_n is bounded and decreasing, it converges by the Monotone Convergence Theorem. Since both (a_n), (b_n) converge, we also know that $(a_n b_n)$ converges by a homework exercise.

4. (15 points each) Prove or disprove the following:[1]

(a) A sequence (x_n) converges to x if and only if for all $\varepsilon > 0$ and for all $N \in \mathbb{N}$, $n \geq N$ implies

$$|x_n - x| < \varepsilon.$$

(b) A sequence (x_n) converges if and only if for all $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $n \geq N$ implies

$$|x_n - x_{n+1}| < \varepsilon.$$

Solutions:

(a) This is false because the quantifier "for all N"cares about even the number $N = 1$. But we don't care about the first finitely many terms of the sequence. This leads to a simple counterexample: let (x_n) be the sequence $(1, 0, 0, 0, ...)$, which we know converges to $x = 0$. However, if $N = 1$, then we have $1 = n \geq N$, but

$$|x_1 - x| = |1 - 0| = 1 > \varepsilon$$

for all positive $0 < \varepsilon < 1$. To summarize, we found a convergent sequence, some ε, and some N such that the necessary inequality doesn't hold, and therefore this cannot be an equivalent definition of convergence.

(b) This looks like a slightly modified version of Cauchy convergence and might be one of the hardest counterexamples to come up with so far.

We talked a lot about the divergence of the harmonic series $\sum 1/m$. Notice that if we consider the partial sums $s_n = \sum_{m=1}^{n} 1/m$, then these partial sums satisfy

$$|s_{n+1} - s_n| = \frac{1}{n+1}.$$

We also know that $s_n \to \infty$, and therefore does not converge to any real number. Since for all $\varepsilon > 0$ there is some n such that $1/(n+1) < \varepsilon$, these partial sums satisfy the new modified Cauchy definition. Therefore this definition is not equivalent to convergence.

[1] Recall that a statement can be disproved by providing one counterexample.

5. Let $\sum_{n=1}^{\infty} a_n$ be a convergent infinite series.[2] For each n, define two new sequences by:

$$p_n = \begin{cases} a_n & \text{if } a_n > 0 \\ 0 & \text{otherwise,} \end{cases} \qquad q_n = \begin{cases} a_n & \text{if } a_n < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Note that all $p_n \geq 0$ but all $q_n \leq 0$, and at least one of p_n, q_n must be zero for all n.

(a) (10 points) Argue that the sequences (p_n) and (q_n) both converge to zero. (Hint: $a_n = p_n + q_n$.)

(b) (15 points) Prove that if $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ both converge.

(c) (Extra credit, 10 points) Prove that if $\sum_{n=1}^{\infty} a_n$ converges conditionally, then $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ both diverge.

Solution:

(a) Since $\sum a_n$ converges, we know $a_n \to 0$. Since for all n at least one of p_n, q_n is zero, we actually know

$$|a_n| = |p_n + q_n| = \max\{|p_n|, |q_n|\} = p_n - q_n = |p_n| + |q_n|.$$

(We will only use one of these equivalent ways of writing $|a_n|$ below.) Therefore

$$0 \leq |p_n| \leq |p_n| + |q_n| = |p_n + q_n| = |a_n|.$$

So by the squeeze theorem, $p_n \to 0$. The same argument works by switching p and q.

(b) This follows by the comparison test. We know that $0 \leq |p_n| \leq |a_n|$, and $\sum |a_n|$ converges, therefore $\sum |p_n|$ converges, and therefore $\sum p_n$ converges absolutely. The same argument works for q.

(c) If $\sum a_n$ converges conditionally, this means $\sum |a_n|$ diverges. But $|a_n| = p_n - q_n$ implies that

$$\sum |a_n| = \sum p_n - \sum q_n.$$

If p_n and q_n both converged, by a theorem from class, we know $\sum |a_n|$ must also converge. Therefore at least one of $\sum p_n, \sum q_n$ diverges. Since p_n are all positive, $\sum p_n$ diverges means $\sum p_n = \infty$. Since q_n are all negative, that means if $\sum q_n$ diverges then $\sum q_n = -\infty$.

Since $\sum a_j$ converges, we know that there is N such that $n > m > N$ implies

$$\left| \sum_{j=m+1}^{n} a_j \right| < 1,$$

and unraveling the definition of absolute value gets us

$$-1 < \sum_{j=m+1}^{n} a_j < 1.$$

[2] If you are having a hard time, write these definitions out for the example $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

By definition of p_j, q_j,

$$-1 < \sum_{j=m+1}^{n} p_j + \sum_{j=m+1}^{n} q_j < 1.$$

Therefore:

$$-1 - \sum_{j=m+1}^{n} q_j < \sum_{j=m+1}^{n} p_j < 1 - \sum_{j=m+1}^{n} q_j \tag{1}$$

and

$$-1 - \sum_{j=m+1}^{n} p_j < \sum_{j=m+1}^{n} q_j < 1 - \sum_{j=m+1}^{n} p_j. \tag{2}$$

If $\sum p_n$ diverges, we can use (2) to show that $\sum q_n = -\infty$. If $\sum q_n$ diverges, we can use (1) to show $\sum p_n = \infty$. Therefore both sums diverge.