MTH 320: Homework 9

The following are due on Monday, March 18:
§4.2:

1. Use the divergence criterion to show that the following limits don't exist:
   (a) $\lim_{x\to 0} |x|/x$,
   (b) $\lim_{x\to 0} \cos(1/x^2)$.
2. (a) Construct a rigorous definition for a limit statement of the form $\lim_{x\to c} f(x) \to \infty$. Use it to prove that $\lim_{x\to 0} 1/x^2 = \infty$.
   (b) Construct a definition for the statement $\lim_{x\to\infty} f(x) = L$.
   (c) Construct a definition for the statement $\lim_{x\to\infty} f(x) = \infty$ and provide an example of such a function.
3. **(Squeeze Theorem)**: Let f, g, h satisfy $f(x) \leq g(x) \leq h(x)$ for all x in some common domain A. If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} h(x) = L$, show that also $\lim_{x\to c} g(x) = L$.
4. Prove a squeeze theorem like above, except replace c with ∞, and use your definition from 2(b).

§4.3:

4. Prove that $g(x) = \sqrt[3]{x}$ is continuous on the real line. (Hint: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$.)
5. (a) Prove any function with domain $\mathbb{Z}$ is continuous everywhere (i.e., everywhere in its domain).
   (b) Prove that in general, given a function $f : A \to \mathbb{R}$, if c is an isolated point of A, then f is continuous at c
6. **(Contraction mapping Theorem)** Let f be defined on all of $\mathbb{R}$. Assume there is a constant $0 < c < 1$ such that

$$|f(x) - f(y)| \leq c|x - y|$$

   for all $x, y, \in \mathbb{R}$.

   (a) Show that f is continuous.
   (b) Pick any point $y_1 \in \mathbb{R}$. Consider the sequence

$$(y_1, f(y_1), f(f(y_1)), \ldots),$$

   i.e., the sequence is defined by: the first term is y_1, and the n^{th} term y_n is defined to be $f(y_{n-1})$.
   Prove (y_n) is Cauchy. (Therefore it makes sense to consider $y = \lim y_n$.)

(c) Prove that y is a fixed point of f ($f(y) = y$).

(d) Prove y is the unique fixed point of f by showing for any x, the sequence

$$(x, f(x), f(f(x)), \dots)$$

converges to y.

(e) (Not graded; just to think about): Can we solve for y? Where does the proof fail if $c = 1$? Does the proof still fail if we replace our condition with $|f(x) - f(y)| < |x - y|$?

7. Prove Thomae's function is discontinuous at all rational points, and is continuous at all irrational points. Thomae's functions is defined by

$$f(x) = \begin{cases} 1 & if x = 0, \\ 1/n & if x = m/n \text{ where } n, m \text{ don't share a common factor,} \\ 0 & if x \text{ is irrational.} \end{cases}$$