MTH 320: Homework 8

The following are due on Monday, March 11:
§3.3:

1. If K is compact, prove $\sup K$ and $\inf K$ both exist and are elements of K.
2. Show if K is compact and F is closed, then $K \cap F$ is compact.
3. Prove that a perfect set is uncountable (hint: the proof is in the book).

§3.4:

4. If P is a perfect set and K is compact, is $P \cap K$ perfect? Compact?
5. A set E is totally disconnected if given any two points $x, y \in E$, there are separated sets A, B with $x \in A$, $y \in B$ and $E = A \cup B$.
   (a) Show $\mathbb{Q}$ is totally disconnected.
   (b) Show that the set of irrational numbers is totally disconnected.
6. You proved on a previous homework that the Cantor set C does not contain any open intervals. Use this to show that C is totally disconnected. (Therefore C is an uncountable, compact set without isolated points that is totally disconnected, and has "length"zero. It's also an example of a fractal. There are still many important open questions about these kinds of sets. In general, we know very little about arbitrary subsets of real numbers.)

§3.5: We say a set $D \subset \mathbb{R}$ is *dense* in $\mathbb{R}$ if $\bar{D} = \mathbb{R}$. We say a set $E \subset \mathbb{R}$ is *nowhere dense* in $\mathbb{R}$ if $\bar{E}$ does not contain any (nonempty) open intervals.

7. Prove that E is nowhere dense in $\mathbb{R}$ if and only if the complement $\bar{E}$ is dense in $\mathbb{R}$.
8. Decide (AKA prove) whether the following sets are dense in $\mathbb{R}$, nowhere dense, or somewhere inbetween (for example, (a) is dense in $[0, 5]$).
   (a) $\mathbb{Q} \cap [0, 5]$.
   (b) $\{1/n : n \in \mathbb{N}\}$.
   (c) The set of irrational numbers.
   (d) The Cantor set.