MTH 320: Homework 5

The following are due on Monday, February 11:
§2.7:

1. (Alternating series test) Let (a_n) be a sequence satisfying

   (i) $a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots$ and
   (ii) $a_n \to 0$.

   Then, the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges. Hint: this amounts to showing that the sequence of partial sums $s_m = a_1 - a_2 + \cdots + (-1)^{m+1}a_m$ converges. Different characterizations of the completeness of $\mathbb{R}$ lead to different proofs:

   (a) Prove the alternating series test by showing (s_m) is a Cauchy sequence.
   (b) Prove the alternating series test using the Nested Interval Property.
   (c) Prove the alternating series test by considering subsequences (s_{2m}) and (s_{2m+1}), and using Monotone Convergence Theorem.

2. (Ratio test) Given a series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$, the ratio test says that if (a_n) satisfies
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = r < 1,$$
   then the series converges absolutely. (You can actually replace the limit with a limsup.)

   (a) Let t satisfy $r < t < 1$ (why does such an t exist?). Explain why there is some $N \in \mathbb{N}$ such that $n \geq N$ implies $|a_{n+1}| < |a_n|t$.
   (b) Why does $|a_N|\sum_{n=1}^{\infty} t^n$ converge?
   (c) Using the above, show that $\sum_{n=1}^{\infty}|a_n|$ converges.

3. (Summation by parts) Let (x_n) and (y_n) be arbitrary sequences and let $s_n = x_1 + \cdots + x + n$. Use the observation that $x_j = s_j - s_{j-1}$ to verify the formula
$$\sum_{j=m+1}^{n} x_j y_j = s_n y_{n+1} - s_m y_{m+1} + \sum_{j=m+1}^{n} s_j(y_j - y_{j+1}).$$
   How does summation by parts relate to integration by parts? What are the similarities and differences?

4. Study for your exam, which is on Friday, February 15. There are many examples and counterexamples in the exercises of this section which I will want to know about on the exam. I might give these as true/false questions with explanation required. Other examples from previous sections would also be helpful to know.