MTH 320: Homework 4

The following are due on Monday, February 4:
§2.4:

1. Let (a_n) be a bounded sequence.
   (a) Prove that the sequence defined by $y_n = \sup\{a_k : k \geq n\}$ converges. (First try some examples with sequences you know, so you get a feel for what's going on.)
   (b) The *limit superior* of (a_n), or $\limsup a_n$ is defined by
   $$\limsup a_n = \lim y_n,$$
   where y_n was defined in (a). Provide a reasonable definition of $\liminf a_n$ and briefly explain why it always exists for any bounded sequence.
   (c) Prove that $\liminf a_n \leq \limsup a_n$ and find an example of a sequence for which the equality is strict.
   (d) Show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. Show that in this case, all three are the same value.

§2.5:

2. If (a_n) and (b_n) are Cauchy, prove that (c_n) is Cauchy, where c_n is defined by $c_n = |a_n - b_n|$.

3. Prove that (a_n) converges to $a \in \mathbb{R}$ iff all subsequences (a_{n_k}) of (a_n) converge to a. Do the same for $a = \infty$.

4. A different proof of Bolzano-Weierstrass: let (a_n) be a bounded sequence. Define the set
   $$S = \{x \in \mathbb{R} : x > a_n \text{ for infinitely many } n\}.$$
   Show that there exists a subsequences (a_{n_k}) converging to $s = \inf S$.

§2.6

5. Give an example of each of the following, or prove that no example exists.
   (a) A Cauchy sequence that is not monotone.
   (b) A monotone sequence that is not Cauchy.
   (c) A Cauchy sequence with a divergent subsequence.
   (d) An unbounded sequence containing a subsequence that is Cauchy.

5. Let (a_n) be a sequence with the property that: for all $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $n \geq N$ implies $|a_{n+1} - a_n| < \varepsilon$. Either prove that (a_n) is Cauchy or else find a counterexample.

§2.7

6. Find an example of a_n, b_n such that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both diverge, but $\sum_{n=1}^{\infty} a_n b_n$ converges.

7. (a) Find an example of a convergent series $\sum_{n=1}^{\infty} a_n$ and a bounded sequence (b_n) such that $\sum_{n=1}^{\infty} a_n b_n$ diverges.

   (b) Show that if in addition in part (a) we assume that $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n b_n$ must converge.