MTH 320: Homework 3

Notation: $\lim a_n = a$ is the same as $\lim_{n\to\infty} a_n = a$ is the same as $(a_n) \to a$ is the same as $a_n \to a$.

The following are due on Monday, January 28:

§2.2:

0. Use the definition of limit to prove the constant sequence $(c, c, ...)$ converges to c.

1. Use the definition of limit to prove $\displaystyle\lim_{n\to\infty} \frac{1}{6n^2+1} = 0$.

2. Use the definition of limit to prove $\displaystyle\lim_{n\to\infty} \frac{2n+3}{5n+7} = \frac{2}{5}$.

3. Prove $(1, 0, 0, 1, 0, 0, 1, 0, 0, ...)$ does not have a limit.

4. *(Practice with quantifiers.)*

   We say (just for this exercise) a sequence (a_n) is ***eventually in*** a set $A \subset \mathbb{R}$ if there exists an $N \in \mathbb{N}$ such that $a_n \in A$ for all $n \geq N$.

   We also say a sequence (a_n) is ***frequently in*** a set $A \subset \mathbb{R}$ if for all $N \in \mathbb{N}$ there is $n \geq N$ such that $a_n \in A$.

   (a) Is the sequence $\{(-1)^n\}$ eventually or frequently in $\{1\}$?

   (b) Does frequently imply eventually or does eventually imply frequently? (Which condition is stronger, i.e., more restrictive.)

   (c) Suppose an infinite number of terms in (x_n) are equal to 4. Is (x_n) eventually in (3.9,4.1)? Is it frequently in $(3.9, 4.1)$?

§2.3:

5. (a) Assume $\lim a_n = a$ and $\lim b_n = b$. Prove that $\lim a_n b_n = ab$.

   (b) Find sequences $(a_n), (b_n)$ that do not converge, but such that $(a_n b_n)$ converges.

6. Assume that $(1/b_n)$ converges. Can you formulate a theorem similar to Theorem 2.3.2? That is, if $1/b_n$ converges, then b_n ________. Prove your theorem.

7. Prove the Squeeze Theorem for limits: if $y_n \leq x_n \leq z_n$, and if $\lim y_n = L = \lim z_n$, then $\lim x_n = L$.

8. Prove that sequences have a unique limit. That is, if $x_n \to x$ and $x_n \to y$, then $x = y$. (You will need a theorem from chapter 1.)

9. If $b_n \to b$, prove that $|b_n| \to |b|$.

§2.4:

10. Show that $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, ...$ converges and find its limit.

11. (a) Prove that the sequence defined by $x_1 = 3$ and $x_{n+1} = \dfrac{1}{4 - x_n}$ for $n > 1$ converges.

    (b) Explain why $\lim x_n$ exists implies $\lim x_{n+1}$ exists. (Maybe by using definition of limit.)

    (c) Take the limit of both sides of the equation in part (a) to explicitly compute $\lim x_n$. Explain why you are allowed to "move"the limits wherever you do.

12. Let $x_1 = 2$ and define $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{2}{x_n}\right)$ for $n > 1$.

    (a) Show that x_n^2 is always greater than 2, and use this to prove $x_n - x_{n+1} \geq 0$. Conclude $\lim x_n = \sqrt{2}$.

    (b) Modify the sequence (x_n) so that it converges to $\sqrt{c}$ for any $c > 0$.

    The above exercise is a good way to approximate square roots with a computer, and it was known by the Babylonians in 1500BC, and by the Greeks in the early 1st century.