MTH 320: Homework 12

The following are due on Wednesday, April 24:
§6.2:

1. Let
$$f_n(x) = \frac{nx}{1+nx^2}.$$
   (a) Find the pointwise limit of (f_n) on $(0, \infty)$.
   (b) Is the convergence uniform on $(0, \infty)$?
   (c) Is the convergence uniform on $(0, 1)$?
   (d) Is the convergence uniform on $(1, \infty)$?

2. Let
$$g_n(x) = \frac{x}{1+x^n}.$$
   (a) Find the pointwise limit of (g_n) on $[0, \infty)$.
   (b) Why is the convergence not uniform?

3. Let
$$h_n = \begin{cases} 1 & \text{if } |x| \geq 1/n \\ n|x| & \text{if } |x| < 1/n. \end{cases}$$
   Find the pointwise limit of (h_n) on $\mathbb{R}$ and decide whether or not the convergence is uniform.

§6.3:

4. Consider the sequence of functions
$$f_n(x) = \frac{x^n}{n}.$$
   (a) Show (f_n) converges uniformly on $[0, 1]$ and find $f = \lim f_n$. Show f is differentiable.
   (b) Show that f'_n converges on $[0, 1]$. Is the convergence uniform? Let $g = \lim f'_n$. Is $g = f'$?

§6.4:

5. Prove that if $\sum_{n=0}^{\infty} f_n$ converges uniformly, then (f_n) converges to the zero function.

6. Prove $g(x) = \sum_{n=0}^{\infty} \cos(2^n x)/2^n$ is continuous on $\mathbb{R}$.

7. Prove $h(x) = \sum_{n=0}^{\infty} x^n/n^2$ is continuous on $[-1, 1]$.

§6.5:

8. Use Theorem 6.5.7 to argue that power series are unique. That is, if

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$$

for all $x \in (-R, R)$, then $a_n = b_n$ for all n. Hint: use induction.

9. (Ratio test for series.) Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with $a_n \neq 0$. Assume

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

   (a) Show that if $L \neq 0$, the series converges for all $x \in (-1/L, 1/L)$. (See the Ratio test exercise in 2.7 for a hint.)

   (b) Show that if $L = 0$, then the series converges for all $x \in \mathbb{R}$.