MTH 320: Homework 11

The following are due on Monday, April 15:
§5.2:

1. (Quotient Rule)
   (a) Find a formula for the derivative of $f(x) = 1/x$ (wherever it's differentiable) using the definition of derivative.
   (b) Use the product rule and the chain rule, together with (a), to find the derivative of (f/g).

2. Let
$$f_a(x) = \begin{cases} x^a & \text{whenever } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$
   (a) For which a is f continuous at zero?
   (b) For which a is f differentiable at zero?
   (c) For which a is f twice differentiable at zero?

3. Let $f : (a, b) \to \mathbb{R}$ be differentiable. Prove that if $f'(x)$ is not a constant function, then $f'(c)$ is irrational for some $c \in (a, b)$.

§5.3:

4. Recall from a previous homework that f is contractive on a set A if there exists $0 < c < 1$ such that
$$|f(x) - f(y)| \leq c|x - y|$$
for all $x, y \in A$. Show that if f is differentiable, and if f' is continuous and satisfies $|f'(x)| < 1$ on $[a, b]$, then f is contractive on $[a, b]$.

5. Recall that a fixed point of $f : A \to \mathbb{R}$ is a point $x \in A$ such that $f(x) = x$. Show that if f is differentiable on an interval with $f'(x) \neq 1$, then f has at most one fixed point in that interval.

6. Let h be differentiable on $[0, 3]$, and assume $h(0) = 1$, $h(1) = 2$, $h(3) = 2$.
   (a) Argue that $h(x)$ has a fixed point in $[0, 3]$.
   (b) Prove there is $c \in [0, 3]$ such that $h'(c) = 1/3$.
   (c) Prove there is $c \in [0, 3]$ such that $h'(c) = 1/4$.

7. For part (b) below, $g : \mathbb{R} \to \mathbb{R}$ is defined by
$$g(x) = \begin{cases} x/2 + x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

(a) Assume f is differentiable on (a, b). Prove f is increasing on (a, b) iff $f'(x) \geq 0$ for all $x \in (a, b)$.[1]

(b) Prove that $g(x)$ is differentiable everywhere and satisfies $g'(0) > 0$. Despite this fact, prove that g is *not* increasing on any interval containing 0.

8. (Compare to 7.) Assume $g : (a, b) \to \mathbb{R}$ is differentiable at c. If $g'(c) \neq 0$, show that there is an $r > 0$ such that the ball of radius r around c, $B_r(c) \subseteq (a, b)$, for which $g(x) \neq g(c)$ for all $x \in B_r(c) \setminus \{c\}$.

[1]Recall: $f : (a, b) \to \mathbb{R}$ is increasing on (a, b) if $f(x) \leq f(y)$ for all $a < x < y < b$.