MTH 320: Homework 10

The following are due on Friday, March 29:
§4.4:

1. Show $1/x^2$ is uniformly continuous on $[1, \infty)$ but not on $(0, 1)$.

2. Show that x^3 is continuous on all of $\mathbb{R}$, but not uniformly continuous on all of $\mathbb{R}$.

3. Even though we know its derivative blows up at zero, prove that $x^{1/3}$ is uniformly continuous on all of $\mathbb{R}$ by showing it's uniformly continuous on $[-1, 1]$, $(-\infty, -1]$, and $[1, \infty)$.

   (How does this argument show it's uniformly continuous on all of $\mathbb{R}$?)

4. Let g be defined on all of $\mathbb{R}$. If A is a subset of $\mathbb{R}$, the inverse image of A (with respect to g) is defined to be

   $$g^{-1}(A) = \{x \in \mathbb{R} : g(x) \in A\}.$$

   Prove g is continuous if and only if $g^{-1}(O)$ is open whenever $O \subset \mathbb{R}$ is open.

5. Let (x_n) be a Cauchy sequence. Prove that if f is uniformly continuous on $\mathbb{R}$, then $(f(x_n))$ is also Cauchy.

   Then, find a counterexample if f is only continuous.

§4.5:

6. True or false (requires explanation):

   (a) Continuous functions take bounded open intervals to bounded open intervals.

   (b) Continuous functions take open intervals to open intervals.

   (c) Continuous functions take bounded closed intervals to bounded closed intervals.

7. Is there a continuous function $f : \mathbb{R} \to \mathbb{R}$ with range $\mathbb{Q}$?

8. Assume f is increasing, that is, assume $x < y$ implies $f(x) \leq f(y)$. If f satisfies the Intermediate Value Property, prove f is continuous.

9. Let f be continuous on $[0, 1]$ and assume its range is $[0, 1]$. Prove that f has a fixed point, that is, there is some x such that $f(x) = x$. (Hint: consider $g(x) = f(x) - x$. Either $g(0) = 0, g(1) = 1$, or else apply an important result from this section.)

§4.6:

Definition: We say f is $\alpha-$continuous at x if there is $\delta > 0$ such that

$$y, z \in (x - \delta, x + \delta) \implies |f(x) - f(y)| < \alpha.$$

Definition: Given a function $f : \mathbb{R} \to \mathbb{R}$, let

$$D_\alpha = \{x \in \mathbb{R} : f \text{ is not } \alpha - \text{continuous at } x\}.$$

Also, define

$$D_f = \{x \in \mathbb{R} : f \text{ is not continuous at } x\}.$$

The following exercises provide a proof that the set of continuities of f must be an F_σ set (the countable union of closed sets).

10. Prove $\alpha < \beta$ implies $D_\beta \subseteq D_\alpha$.
11. Prove if f is continuous at x, then it is $\alpha-$continuous at x for all α. Explain how this proves $D_\alpha \subset D_f$ for all α.
12. Prove $D_{1/n}$ is closed.
13. Prove if f is not continuous at x, then f is not $\alpha-$continuous at x for some α. Explain why this implies

$$D_f = \bigcup_{n=1}^{\infty} D_{1/n}.$$

This completes the proof that the set of discontinuities of $f : \mathbb{R} \to \mathbb{R}$ is an F_σ set.[1]

[1] Conversely, given any F_σ set, one can construct a function that is discontinuous on that set. The construction is a more general version of the one for Thomae's function.