1. (20 points)

   (i) Compute the volume of a ball of radius R using Cavalieri's principle. (Hint: consider $x^2+y^2+a^2=1$ as a varies between $-1 \leq a \leq 1$.)

   (ii) Does Cavalieri's principle hold for surface areas? (Hint: the surface area of a sphere of radius R is $4\pi R^2$. You could attempt to apply this method to cross-sections of this sphere.)

Solutions:

(i) If $x^2+y^2+z^2 \leq R^2$, we can take cross sections at z =constant. The intersection is the disk $x^2+y^2 \leq R^2-z^2$. This disk has radius $\sqrt{R^2-z^2}$. The area of the disk is $\pi\sqrt{R^2-z^2}^2 = \pi(R^2-z^2)$. Cavalieri's principle says that the volume of the ball is equal to the integral of these cross sections, so it's equal to

$$\int_{-R}^{R} \pi(R^2-z^2)dz = \pi(R^2 z - z^3/3)\Big|_{-R}^{R} = \pi[(R^3-R^3/3)-(-R+R^3/3)] = \frac{4\pi R^3}{3}.$$

(ii) If Cavalieri's principle worked for surface areas, then we better have that surface area is the integral of the cross sectional perimeters. But the integral of cross sections is

$$\int_{-R}^{R} 2\pi\sqrt{R^2-z^2}dz = 2\pi\int_{-R}^{R}\sqrt{R^2-z^2}dz.$$

The integral just represents the area of the upper half disk of radius R, which has area $\pi R^2/2$. Multiplying by 2π, the integral of cross sectional perimeters is $\pi^2 R^2$. This is not equal to $4\pi R^2$. (But it would be equal if the proof of $\pi = 4$ we did in class was true.)

2. (20 points) Compute the following integrals.

 (i) $\int_0^1 \int_y^1 e^{-x^2} dxdy$

 (ii) $\int_0^1 \int_{\sqrt[4]{x}}^1 \int_0^3 z^2 \cos\left(\frac{\pi x}{2y^4}\right) dzdydx$

Solutions: Use Fubini's Theorem. Draw the region.

(i)

$$\int_0^1 \int_y^1 e^{-x^2} dxdy = \int_0^1 \int_0^x e^{-x^2} dydx = \int_0^1 xe^{-x^2} dx = -\frac{e^{-x^2}}{2}\Bigg|_0^1 = -\frac{1}{2e} + \frac{1}{2}.$$

(ii)

$$\begin{aligned}
&\int_0^1 \int_{\sqrt[4]{x}}^1 \int_0^3 z^2 \cos\left(\frac{\pi x}{2y^4}\right) dzdydx \\
=&\int_0^1 \int_{\sqrt[4]{x}}^1 \frac{z^3}{3}\Bigg|_0^3 \cos\left(\frac{\pi x}{2y^4}\right) dydx \\
=&\int_0^1 \int_{\sqrt[4]{x}}^1 9\cos\left(\frac{\pi x}{2y^4}\right) dydx \\
=&9\int_0^1 \int_0^{y^4} 9\cos\left(\frac{\pi x}{2y^4}\right) dxdy \\
=&9\int_0^1 \sin\left(\frac{\pi x}{2y^4}\right)\left(\frac{2y^4}{\pi}\right)\Bigg|_0^{y^4} dx \\
=&\frac{18}{\pi}\int_0^1 (\sin\left(\frac{\pi}{2}\right)\left(\frac{2y^4}{\pi}\right) - 0)dy \\
=&\frac{18}{\pi}\int_0^1 y^4 dy = \frac{18}{5\pi}.
\end{aligned}$$

3. (10 points) Bound the following integral above and below: $\int_0^{10}\int_0^{10}\int_0^{10}\sin(xy^2z^3)dxdydz$.

   Are you happy with this bound? Why?

   Solution:

   We know that $-1 \leq \sin(xy^2z^3) \leq 1$. By monotonicity of integrals,

$$-10^3 = \int_0^{10}\int_0^{10}\int_0^{10} -1dxdydz \leq \int_0^{10}\int_0^{10}\int_0^{10}\sin(xy^2z^3)dxdydz \leq \int_0^{10}\int_0^{10}\int_0^{10} 1dxdydz = 10^3.$$

   So

$$-1000 \leq \int_0^{10}\int_0^{10}\int_0^{10}\sin(xy^2z^3)dxdydz \leq 1000.$$

   This is not a good bound because sin oscillates a lot in the given region. We expect a lot of cancellation.