Given a differentiable function $F : \mathbb{R}^3 \to \mathbb{R}$, its level sets are sufficiently nice surfaces. Using the same argument as yesterday, one can look at (collections of) curves on these surfaces and show that the gradient of such functions is orthogonal to the level sets. In other words, its orthogonal to tangent planes:

$$\nabla F(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0.$$

1. (20 points) Using the above fact, derive the equation of the tangent plane to a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ at $(x_0, y_0, f(x_0, y_0))$.

   Hint: consider a level set of the function $F(x, y, z) = z - f(x, y)$.

   Solution: Let $F(x, y, z) = z - f(x, y)$. Consider the level set $F(x, y, z) = 0$, or equivalently, when $z = f(x, y)$. By the above fact,

   $$\nabla F(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0.$$

   But $\nabla F(x_0, y_0, z_0) = \left(-\frac{\partial f}{\partial x}(x_0, y_0), -\frac{\partial f}{\partial y}(x_0, y_0), 1\right)$,[1] and $z_0 = f(x_0, y_0)$. Plugging this in, we see that

   $$\begin{aligned} 0 &= \left(-\frac{\partial f}{\partial x}(x_0, y_0), -\frac{\partial f}{\partial y}(x_0, y_0), 1\right) \cdot (x - x_0, y - y_0, z - f(x_0, y_0)) \\ &= -\frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) - \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) + z - f(x_0, y_0), \\ \text{which implies } z &= \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) + f(x_0, y_0). \end{aligned}$$

   This is exactly the equation of the tangent plane we saw before.

[1] Which way is this vector pointing? This could be another way to define functions of (x, y).

Assume for question 2 that whenever a and b are constants, f satisfies

$$\frac{d}{dx}\int_a^b f(x,t)dt = \int_a^b \frac{\partial f}{\partial x}(x,t)dt.$$

2. (30 points) *Using the Chain Rule*, find

$$\frac{d}{dx}\int_0^{b(x)} f(x,t)dt.$$

Can you extend this result replacing the lower limit, zero, with $a(x)$?

Hint: This is the $x-$ derivative of some function $G(c(x), b(x))$.

Solution: Let

$$G(c(x), b(x)) = \int_0^{b(x)} f(c(x),t)dt.$$

The chain rule tells us

$$\frac{d}{dx}G(c(x), b(x)) = \frac{\partial G}{\partial c}\frac{dc}{dx} + \frac{\partial G}{\partial b}\frac{db}{dx}.$$

By the fact above, we see

$$\frac{\partial G}{\partial c} = \frac{\partial}{\partial c}\int_0^{b(x)} f(c(x),t)dt = \int_0^{b(x)} \frac{\partial f}{\partial c}(c(x),t)dt,$$

and by the Fundamental Theorem of Calculus,

$$\frac{\partial G}{\partial b} = \frac{\partial}{\partial b}\int_0^{b(x)} f(c(x),t)dt = f(c(x), b(x)).$$

Plugging in $c(x) = x$ we see that $\frac{dc}{dx} = 1$ and so

$$\frac{\partial G}{\partial c}\frac{dc}{dx} + \frac{\partial G}{\partial b}\frac{db}{dx} = \int_0^{b(x)} \frac{\partial f}{\partial x}(x,t)dt + f(x,b(x))b'(x).$$

To generalize this result, consider

$$\int_{a(x)}^{b(x)} f(x,t)dt = \int_{a(x)}^{0} f(x,t)dt + \int_0^{b(x)} f(x,t)dt = -\int_0^{a(x)} f(x,t)dt + \int_0^{b(x)} f(x,t)dt.$$

Applying the work above to these two integrals, we get what's called *Leibniz's integral rule:*

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)dt = f(x,b(x))b'(x) - f(x,a(x))a'(x) + \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x}(x,t)dt.$$

This is a more general version of the Fundamental Theorem of Calculus.