1. Let $\mathbf{u} = (1, 2, 3)$ and $\mathbf{v} = (3, -2, 1)$.

   (a) Find $\mathbf{u} \cdot \mathbf{v}$.

   (b) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

   (c) Find $\mathbf{u} \times \mathbf{v}$.

   (d) What are $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v})$ and $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v})$? Would the expressions still make sense without parentheses? Why or why not?

Solution:

(a)

$$1 \cdot 3 + 2 \cdot (-2) + 3 \cdot 1 = 2.$$

(b)

$$2 = \|\mathbf{u}\|\|\mathbf{v}\| \cos(\theta) = \sqrt{1+4+9}\sqrt{9+4+1}\cos(\theta) = 14\cos(\theta).$$

So $\cos(\theta) = 1/7$, which is the same as $\theta = arccos(1/7)$.

(c)

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 3 & -2 & 1 \end{vmatrix} = (2 - (-6), 9 - 1, -2 - 6) = (8, 8, -8).$$

(d) Since $\mathbf{u} \times \mathbf{v}$ is always orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, both quantities are the scalar zero. These expressions would make sense without parentheses because $\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}$ can only be read one way: we must do the cross product first. If you did the dot product first, it would give a scalar, and then you would have the cross product of a scalar with a vector, which we cannot do: $(\mathbf{u} \cdot \mathbf{v}) \times \mathbf{w}$ doesn't make sense.

You can "double check" that both are zero by doing the computation.

Two matrices can be added by adding their components separately. For example,

$$\begin{bmatrix} 2 & 4 \\ 8 & 16 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 12 & 17 \end{bmatrix}.$$

For the following question, consider 2×2 matrices.

2. Denote the determinant of a matrix A by $\det(A)$.

   (a) Is $\det(A + B) = \det(A) + \det(B)$? Prove it if it's true, or find a counterexample (one case where it's wrong) if it's false. What about $\det(AB) = \det(A)\det(B)$?

   (b) Define a dot product for matrices, and a norm (magnitude) for matrices. How can we think of a matrix as a vector? Can you define an angle between the two matrices? Do you think Cauchy-Schwarz and Triangle Inequality would still hold for this dot product and norm?

Solution:

(a) I didn't really expect you to come up with good proofs/examples on the spot, it was just something to think about.

The first question is false because the determinant of both

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

both have determinant equal to 1. However,

$$\det(A + B) = 0 \neq 1 + 1 = \det(A) + \det(B).$$

With multiplication it's true, but we won't prove it.

(b) We can think of an $m \times n$ matrix as a vector in $\mathbb{R}^{m \times n}$, and in particular a 2×2 matrix as a vector in $\mathbb{R}^4$ by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a, b, c, d).$$

Now we can define the dot product of two matrices and norm of a matrix how we did for vectors. We can even define cross product of two matrices that have a zero component in the same place by thinking of a vector $(a, b, c, 0)$, for example, as the vector (a, b, c) in $\mathbb{R}^3$. In §1.5 we'll see that all the properties of dot product and norm still hold in any $\mathbb{R}^n$, and so will all the inequalities we proved.