1. (30 points) Let $\mathbf{c}'(t) = (2t, 3t^2, 2/3)$.

(a) Given that $\mathbf{c}(1) = (0, 1, 1)$, find $\mathbf{c}(t)$.

(b) Find the arclength of $\mathbf{c} : [1, 4] \to \mathbb{R}^3$.

(c) Write down $\int_C xyzds$ as some integral $\int_a^b f(t)dt$, where C is the path parametrized by $\mathbf{c} : [1, 4] \to \mathbb{R}^3$.

Solution:

(a) By the FTC,

$$\mathbf{c}(t) - (0, 1, 1) = \mathbf{c}(t) - \mathbf{c}(1) = \int_1^t \mathbf{c}'(u)du = (u^2, u^3, 2u/3)\Big|_1^t = (t^2, t^3, 2t/3) - (1, 1, 2/3).$$

So

$$\mathbf{c}(t) = (t^2, t^3, 2t/3) - (1, 1, 2/3) + (0, 1, 1) = \left(t^2 - 1, t^3, \frac{2t+1}{3}\right).$$

(b) The arclength is

$$\int_1^4 \|\mathbf{c}'(t)\|dt = \int_1^4 \|(2t, 3t^2, 2/3)\|dt = \int_1^4 \sqrt{4t^2 + 9t^4 + 4/9}dt.$$

This problem is either change of variables, or the inside of the square root is a square. If $4t^2+9t^4+4/9 = 9t^4 + 4t^2 + 4/9$ is a square, it can only equal $((9t^4)^{1/2} + (4/9)^{1/2})^2 = (3t^2 + 2/3)^2$. You can check that indeed it does. Therefore

$$\int_1^4 \|\mathbf{c}'(t)\|dt = \int_1^4 3t^2 + 2/3dt = t^3 + 2t/3\Big|_1^4 = (64 + 8/3) - (1 + 2/3) = 65.$$

(c)

$$\int_C xyzds = \int_1^4 (t^2 - 1)(t^3)\left(\frac{2t+1}{3}\right)\sqrt{4t^2 + 9t^4 + 4/9}dt.$$

Notice that part (b), the arclength, is just $\int_C ds$.

2. (25 points)(+5 points) Let $\mathbf{F}(x,y) = (y,-x)$. Recall that when talking about curl of $\mathbf{F}$, we can think of $\mathbf{F}$ as a vector field on $\mathbb{R}^3$ defined by $F(x,y,z) = (y,-x,0)$.

(a) Sketch the vector field $\mathbf{F}$ and a few of its flow lines.

(b) Compute $\nabla \times \mathbf{F}$. Is $\mathbf{F}$ a gradient field?

(c) Compute $\nabla \cdot \mathbf{F}$. Is $\mathbf{F}$ equal to the curl of any vector field?

Solution:

(a) A little too much work to graph here, sorry. You should check at eight points: $(\pm 2, 0), (0, \pm 2), (\pm 1, \pm 1)$.

It is a vector field going clockwise around the origin, with vectors having constant magnitude on circles $x^2 + y^2 = R^2$. This is because on these circles, $\|(y,-x)\| = \sqrt{x^2+y^2} = R$. It is actually because of this reason that the flow lines turn out to be circles. A rigorous way to see that flow lines are cirlces is to consider $\mathbf{c}_R(t) = (R\cos(t), R\sin(t))$ and check all $\mathbf{c} = \mathbf{c}_R$ are solutions to $\mathbf{F}(\mathbf{c}(t)) = \mathbf{c}'(t)$.

(b) $\nabla \times (y,-x,0) =$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & -x & 0 \end{vmatrix} = \left(\frac{\partial 0}{\partial y} - \frac{\partial(-x)}{\partial z}, \frac{\partial y}{\partial z} - \frac{\partial 0}{\partial x}, \frac{\partial(-x)}{\partial x} - \frac{\partial y}{\partial y} \right) = (0,0,-2).$$

Since this is not the zero vector, $\mathbf{F}$ cannot be a gradient field. This is because $\nabla \times \nabla f = (0,0,0)$.

(c)

$$\nabla \cdot (y,-x,0) = 0.$$

The only thing we can say so far is that $(y,-x,0)$ may or may not be the curl of another vector field. We know that the divergence of curl is zero, or equivalently: if the curl is not zero, the vector field is not a curl. The only way we know to conclude a vector field is a curl is to explicitly find $\mathbf{G}$ such that $(y,-x,0) = \nabla \times \mathbf{G}$.

Theorem 8 of section 8.3 (p.459) says that if $\mathbf{F}$ is defined everywhere and the divergence is zero, it must be a curl. One can check that $(y,-x,0)$ is the curl of $\mathbf{G} = (1/2)(0,0,x^2+y^2)$, and many other vector fields. If you set up a system of equations, and chose one of the components to be zero, you can find many other ones. Notice that $\mathbf{G}$ is related to circles as well.

3. (45 points) In no particular order, solve these integration problems. Any four correct solutions are out of 10 points each, the last one is 5 points.

(a) Find the volume of the region in the first octant $(x, y, z \geq 0)$ bounded by $3x + 2y + z = 1$.

(b) Let B be the ball of radius one in $\mathbb{R}^3$ centered at $(0, 0, 0)$ and define f_λ on $B \backslash \{(0, 0, 0)\}$ by

$$f_\lambda(x, y, z) = \frac{1}{(x^2 + y^2 + z^2)^\lambda}.$$

For what λ is f_λ integrable on its domain, $B \backslash \{(0, 0, 0)\}$?

(c) Let D be a triangle in the (x, y) plane with vertices $(0, 0)$, $(1, 1)$, and $(2, 0)$. Compute

$$\iint_D \cos\left(\frac{x - y}{x + y}\right) dA.$$

(d) Find the volume bounded by the sphere of radius 2 centered at the origin and the cylinder $x^2 + y^2 = 1$.

(e) Find the average value *of the radius* of a ball of radius R.

Solutions:

(a) Since we are in the first octant, $0 \leq z \leq 1 - 3x - 2y$. Set these two bounds equal to see that $y = (1 - 3x)/2$ must be a bound for y. Since we are in the first octant, the other bound is 0, so $0 \leq y \leq (1 - 3x)/2$. Finally, repeat the same process to see that $0 \leq x \leq 1/3$.

The correct volume is

$$\int_0^{1/3} \int_0^{(1-3x)/2} \int_0^{1-3x-2y} dz dy dx.$$

The simplest way I know to compute it is

$$\int_0^{1/3} \int_0^{(1-3x)/2} 1 - 3x - 2y dy dx = \int_0^{1/3} (1 - 3x)y - y^2 \Big|_0^{(1-3x)/2} dx$$

$$= \int_0^{1/3} \frac{(1 - 3x)^2}{2} - \frac{(1 - 3x)^2}{4} dx = \int_0^{1/3} \frac{(1 - 3x)^2}{4} dx = \frac{(1 - 3x)^3}{4 \cdot (-3) \cdot 3}\Big|_0^{1/3} = \frac{1}{36}.$$

In general, the volume of a tetrahedron with sidelength a, b, c is $\frac{abc}{6}$. In this case, it was $\frac{1 \cdot (1/2) \cdot (1/3)}{6}$.

(b) Use spherical coordinates to make the integral equal to

$$\int_0^{2\pi} \int_0^{\pi} \int_0^1 \frac{\rho^2 \sin(\phi)}{\rho^{2\lambda}} d\rho d\phi d\theta.$$

Since theta and phi have nothing to do with this problem (split up the integrals into $d\phi d\theta$ and $d\rho$ separately), we need to see when $\int_0^1 \frac{\rho^2}{\rho^{2\lambda}} d\rho$ exists. This is an improper integral equal to

$$\lim_{\varepsilon \to 0^+} \int_\varepsilon^2 \rho^{2-2\lambda} d\rho = \lim_{\varepsilon \to 0^+} \frac{2^{3-2\lambda} - \varepsilon^{3-2\lambda}}{3 - 2\lambda}.$$

whenever $3 \neq 2\lambda$. Now we have to check when the limit $\lim_{\varepsilon \to 0^+} \varepsilon^{3-2\lambda}$ exists. It will exist whenever the exponent is nonnegative, or $3 - 2\lambda \geq 0 \implies \lambda \leq 2/3$.

Therefore the integral exists whenever $\lambda < 2/3$.

(c) Use the change of variables $u = x - y, v = x + y$. Then compute the Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2.$$

Since this is already positive, we don't need to take the absolute value.

We learned that the chain rules implies

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}} = \frac{1}{2}.$$

Therefore, $dxdy = \frac{1}{2}dudv$.

Next, we need to compute the new limits. Since $u = x - y$, $v = x + y$, and we know vertices go to vertices, $(0,0), (1,1), (2,0)$ in the $xy-$plane go to $(0,0), (0,2), (2,2)$ in the $uv-$plane, respectively. Therefore the region is a triangle, a simple region, described by

$$\{(u,v) : 0 \leq u \leq v, 0 \leq v \leq 2\} \text{ or } \{(u,v) : u \leq v \leq 2, 0 \leq u \leq 2\}.$$

Now $\cos \dfrac{x-y}{x+y} = \cos \dfrac{u}{v}$ would be much easier to integrate with respect to u. So

$$\iint_D \cos\left(\frac{x-y}{x+y}\right) dA = \int_0^2 \int_0^v \cos(u/v)\frac{1}{2}dudv = \frac{1}{2}\int_0^2 v\sin(u/v)\Big|_{u=0}^{u=v} dv = \frac{1}{2}\int_0^2 v\sin(1) = \sin(1).$$

(d) In rectangular coordinates, $-\sqrt{4-x^2-y^2} \leq z \leq \sqrt{4-x^2-y^2}$. The cylinder gives us the region in the $xy-$plane: $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}, -1 \leq x \leq 1$. So one way to set up the integral is

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} dzdydx = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\sqrt{4-x^2-y^2}dydx.$$

Now we can switch to polar, noting that the region is just the disk $x^2 + y^2 \leq 1:$

$$= \int_0^{2\pi} \int_0^1 2\sqrt{4-r^2}rdrd\theta = 4\pi \int_0^1 \sqrt{4-r^2}rdr.$$

Now you can let $u = 4 - r^2, du = -2rdr$ to change the integral into

$$4\pi \int_0^1 \sqrt{4-r^2}rdr = -2\pi \int_4^3 \sqrt{u}du = 2\pi \int_3^4 \sqrt{u}du = 2\pi(4^{3/2} - 3^{3/2}) \cdot (2/3).$$

(e) The average value of any function f over a set D is

$$\frac{\iiint_D f dV}{\iiint_D dV} = \frac{1}{\text{Volume of } D} \iiint_D f dV.$$

The volume of a ball of radius R was computed in class to be $4\pi R^3/3$. To find the integral in the numerator, we should use spherical coordinates:

$$\iiint_D f dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \rho \cdot \rho \sin(\phi) d\rho d\phi d\theta = 2\pi \int_0^{\pi} \sin(\phi) d\phi \cdot \int_0^{R} \rho^3 d\rho = 4\pi \frac{R^4}{4}.$$

Dividing by the volume of the ball gives us that the average radius is $3R/4$.

In general (in $\mathbb{R}^n$), the average value is equal to $nR/(n+1)$, which tells us most of the radius is very close to the boundary of the ball. This is *almost* the same as saying most of the volume of the ball is near the boundary, but not quite the same (although most of the ball is near the boundary, it is a different portion).