1. (20 points)

(a) Define f in $\mathbb{R}^2$ by $f(x,y) = x^{1/3}y^{1/3}$. Find $\frac{\partial f}{\partial x}(0,0)$.

If you try taking the derivative using Calc 1, you will get 0/0, which is undefined. So we have to do another method: the definition of derivative.

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0^{1/3}h^{1/3}-0^{1/3}0^{1/3}}{h} = \lim_{h\to 0}\frac{0}{h} = 0.$$

We did this in class – this was one example why partial derivatives don't exactly reflect differentiability the way we want.

(b) Define f in $\mathbb{R}^2$ by $f(x,y) = y\cos(|\sin(x^5)|^2)$. Compute f_{xyxyx}. Justify each step.

We had a theorem that said $f_{xy} = f_{yx}$ as long as f is twice continuously differentiable (C^2). Similarly, if f is C^5, then $f_{xyxyx} = f_{yyxxx}$ (actually we only need C^4 since we only switch the first four derivatives). All that's left is to check f is indeed C^5.

There might be a problem because absolute value is not differentiable. So we can't just use the argument that the composition and multiplication of C^5 functions is C^5.

Luckily, $|\text{whatever}|^2 = \text{whatever}^2$. So $f(x,y) = y\cos(\sin(x^5)^2) = y\cos(\sin^2(x^5))$. Now you can say that this is a composition of C^5 functions multiplied by a C^5 function, and use the theorem.

2. (30 points) Define f and g from $\mathbb{R}^2$ to $\mathbb{R}^2$ by $g(x,y) = (x^2 - y^2, 2xy)$. Let $f(u,v) = (e^u\cos(v), e^u\sin(v))$. Let $h(x,y) = (f\circ g)(x,y)$. Find $Dh(1,1)$.

By the Chain Rule, $Dh(1,1) = Df(g(1,1))Dg(1,1)$.
Since $g(1,1) = (1^2 - 1^2, 2(1)(1)) = (0,2)$, we need to find $Df(0,2)$ and $Dg(1,1)$.

$$Df = \begin{bmatrix} \frac{\partial}{\partial u} e^u\cos(v) & \frac{\partial}{\partial v} e^u\cos(v) \\ \frac{\partial}{\partial u} e^u\sin(v) & \frac{\partial}{\partial v} e^u\sin(v) \end{bmatrix} = \begin{bmatrix} e^u\cos(v) & -e^u\sin(v) \\ e^u\sin(v) & e^u\cos(v) \end{bmatrix},$$

which at $(0,2)$ equals

$$\begin{bmatrix} \cos(2) & -\sin(2) \\ \sin(2) & \cos(2) \end{bmatrix}.$$

Next,

$$Dg = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix},$$

which at $(1,1)$ equals

$$\begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}.$$

Multiplying the two matrices Df with Dg yields

$$\begin{bmatrix} \cos(2) & -\sin(2) \\ \sin(2) & \cos(2) \end{bmatrix}\begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2\cos(2) - 2\sin(2) & -2\cos(2) - 2\sin(2) \\ 2\sin(2) + 2\cos(2) & -2\sin(2) + 2\cos(2) \end{bmatrix}.$$

You can think of $h_x(1,1)$ as the left column of this matrix, and $h_y(1,1)$ as the right column.

3. (30 points) Define f in $\mathbb{R}^2$ by $f(x, y) = 2x^3 + 6xy^2 - 3x^2 + 3y^2$.

(a) Find the critical points of f.
We have to solve for

$$\nabla f = (6x^2 + 6y^2 - 6x, 12xy + 6y) = (0, 0).$$

Therefore $0 = 12xy + 6y = 6y(2x + 1)$. So either $y = 0$ or $x = -1/2$.

If $y = 0$, then $6x^2 + 6y^2 - 6x = 6x^2 - 6x = 6x(x - 1) = 0$ when $x = 0$ or $x = 1$.

If $x = -1/2$, then $6x^2 + 6y^2 - 6x = 1.5 + 6y^2 + 3 = 0$ has no solutions. So the only critical points are $(0, 0)$ and $(1, 0)$.

(b) Find and classify the extrema (out of the choices: local min/ local max/ saddle).
$f_{xx} = 12x - 6, f_{yy} = 12y + 6, f_{xy} = 12y$. Since we are looking only where $y = 0$, let's plug that in first to make it easier. So actually

$$f_{xx}(x, 0) = 12x - 6, f_{yy}(x, 0) = 6, f_{xy}(x, 0) = 0.$$

Therefore the Hessian determinant is equal to $|H| = 72x - 36$. At $x = 0$, it's negative, therefore $(0, 0)$ is a saddle point. At $x = 1$, it's positive. Since $f_{yy} > 0$, we know that $(1, 0)$ is a local minimum.

(c) Find the absolute maximum and minimum on the right half disk: the region D bounded to the right by $x^2 + y^2 = 1$ and to the left by $x = 0$.

Since we have a few y^2 in our function, I think the best parametrizations to take are

(i) $x = 0$, $-1 \leq y \leq 1$;
(ii) $y = \sqrt{1 - x^2}$, $0 \leq x \leq 1$;
(iii) $y = -\sqrt{1 - x^2}$, $0 \leq x \leq 1$.

We already saw the gradient is never zero in the interior of D, so we just have to check the three functions above.

(i) $f(0, y) = 3y^2$, which has one critical point at $y = 0$. The values to keep track of here are $f(0, 0) = \mathbf{0}$, $f(0, 1) = \mathbf{3}$, and $f(0, -1) = \mathbf{3}$.
(ii) Checking the endpoints, $f(0, 0) = \mathbf{0}$ and $f(1, 0) = 2 - 3 =$ **-1**. For the rest, see below.
(iii) Checking the endpoints, $f(0, 0) = \mathbf{0}$ and $f(1, 0) = 2 - 3 =$ **-1**, as above. For the interior $(0, 1)$,[1]

$$f(x, \pm\sqrt{1 - x^2}) = 2x^3 + 6x(1 - x^2) - 3x^2 + 3(1 - x^2) = -4x^3 - 6x^2 + 6x + 3.$$

Call the above $g(x)$. Then $g'(x) = -12x^2 - 12x + 6 = 0$ when $x^2 + x - 1/2 = 0$. Using the quadratic formula, we get

$$x = \frac{-1 \pm \sqrt{3}}{2},$$

[1] Skip this part unless you really like inequalities, and want to learn how to deal with square roots. I didn't mean it to be so messy, I made a typo. Sorry about that.

but only $x = \frac{-1+\sqrt{3}}{2}$ might lie in our interval (and does since $0 < -1+\sqrt{3} < -1+2 = 1$). Since $x^2 + x - 1/2 = 0$,

$$x^2 = 1/2 - x = \frac{2-\sqrt{3}}{2}.$$

Also,

$$x^3 = x \cdot x^2 == \frac{-1+\sqrt{3}}{2}\frac{2-\sqrt{3}}{2} = \frac{-5+3\sqrt{3}}{4}.$$

Finally,

$$g(x) = -4\left(\frac{-5+3\sqrt{3}}{4}\right) - 6\left(\frac{2-\sqrt{3}}{2}\right) + 6\left(\frac{-1+\sqrt{3}}{2}\right) + 3$$
$$= (5-3\sqrt{3}) + (-6+3\sqrt{3}) + (-3+3\sqrt{3}) + 3 = -1+3\sqrt{3}.$$

Now $27 > 16$, so taking the square root of both sides, we see that $3\sqrt{3} > 4$ and therefore $-1+3\sqrt{3} > 3$.

So $-1+3\sqrt{3}$ is the maximum on D and -1 is the minimum.

4. (20 points) Find the second order Taylor Polynomial of $f(x, y) = e^{2xy} + x^2 - 2y^3$ at $(0, 0)$ using any correct method.

Most of you got this right, so let me use a different method. Let P be the function taking any C^2 function f to its Taylor Polynomial of order 2:

$$P(f) = \text{Taylor Polynomial of } f \text{ of order 2.}$$

Fact: P is linear. In particular, $P(f + g) = P(f) + P(g)$. Therefore,

$$P(e^{2xy} + x^2 - 2y^3) = P(e^{2xy}) + P(x^2) + P(-2y^3).$$

The second order expansion of any second degree polynomial is itself, and the second order expansion of a polynomial with no second degree or lower terms is zero:[2]

$$P(x^2) = x^2, \text{ and } P(-2y^3) = 0.$$

Finally, we can use Calc 2 methods to write

$$P(e^{2xy}) = 1 + (2xy) + \frac{(2xy)^2}{2!} + ...$$

Actually this stops at $2xy$ since all higher order terms are higher than second degree (x^2y^2 is fourth degree, and so on). Therefore $P(e^{2xy}) = 1 + 2xy$. Finally,

$$P(e^{2xy} + x^2 - 2y^3) = P(e^{2xy}) + P(x^2) + P(-2y^3) = 1 + 2xy + x^2.$$

This method doesn't always work. For example, if we had $e^{2\sin(xy)}$ instead of e^{2xy}, we wouldn't be able to apply Calc 2 methods as easily.

[2] What does this say about the best possible plane/ paraboloid/etc that approximates third degree polynomials?