Homework 6

The following are due on Friday, October 12:

§4.3 #9, 10[1], 24[2].

§4.4 #1, 21, 33, 37.

§5.1 #3, 5, 7.

§5.2 #12.

Vector field homework problem. We use the heuristic for a C^2 vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ that

$$\nabla \cdot \nabla \times \mathbf{F} = 0$$

because for any vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$,

$$\mathbf{u} \cdot \mathbf{u} \times \mathbf{v} = 0.$$

But obviously $\mathbf{v} \cdot \mathbf{u} \times \mathbf{v} = 0$ holds as well.

Problem:

(a) Show that if the vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ can be expressed as $\mathbf{F} = g\nabla f$ for some C^1 scalar functions $f, g : \mathbb{R}^3 \to \mathbb{R}$, then

$$\mathbf{F} \cdot \nabla \times \mathbf{F} = 0.$$ [3]

(b) If $\mathbf{F} = (F_1(x, y), F_2(x, y), 0)$, show that $\mathbf{F} \cdot \nabla \times \mathbf{F} = 0$.

(c) Find an example of a vector field $\mathbf{F}$ such that $\mathbf{F} \cdot \nabla \times \mathbf{F} \neq 0$.

[1] Justify why for both 9 and 10. Try to find a systematic method that might work for any problem like these.

[2] This is another proof that gradient points in the direction of greatest increase

[3] The converse also holds! Namely, if $\mathbf{F} \cdot \nabla \times \mathbf{F} = 0$, then $\mathbf{F}$ is of this form. However, this is very difficult to prove with what we know so far. Particularly, $\mathbf{F} \cdot \nabla \times \mathbf{F}$ is nonzero in general.