Homework 12 Solutions

8.1.10 Find the area of the disc D of radius R using Green's Theorem.

One formula for the area of a region R with boundary C was

$$\text{Area} = \iint_R dA = \frac{1}{2}\oint_C xdy - ydx.$$

Assume the disk is $(x-a)^2 + (y-b)^2 = R^2$. We can parametrize the boundary counterclockwise by

$$\mathbf{c}(t) = (a + R\cos(t), b + R\sin(t)), 0 \le t \le 2\pi.$$

Then

$$\mathbf{c}'(t) = (-R\sin(t), R\cos(t)) = (x'(t), y'(t)).$$

Therefore

$$xdy - ydx = (a+R\cos(t))(R\cos(t)) - (b+R\sin(t))(-R\sin(t)) = R^2 + aR\cos(t) + bR\sin(t).$$

After parametrizing, the integral becomes

$$\int_0^{2\pi} R^2 + aR\cos(t) + bR\sin(t)dt = 2\pi R^2.$$

Dividing by two gets us the correct area.

8.1.19

(a) Verify the divergence theorem for $\mathbf{F} = (x, y)$ and D the unit disc $x^2 + y^2 \le 1$. We have

$$\iint_D \nabla \cdot \mathbf{F} dA = \iint_D 2dA = 2(\pi 1^2) = 2\pi = \int_0^{2\pi} \cos^2(t) + \sin^2(t)dt$$

$$= \int_0^{2\pi} (\cos(t), \sin(t)) \cdot (\cos(t), \sin(t))\sqrt{\sin^2(t) + \cos^2(t)}dt = \int_C \mathbf{F} \cdot \mathbf{n} ds.$$

We used the fact that $\mathbf{n}$, the unit normal, is just $(\cos(t), \sin(t))$ for any[1] circle.

(b) Evaluate the integral of the normal component $(2xy, -y^2)$ around the ellipse defined by $x^2/a^2 + y^2/b^2 = 1$.

Here we use the divergence theorem to transform the integral into

$$\iint_{\text{ellipse}} \nabla \cdot \mathbf{F} dA = \iint_{\text{ellipse}} 2y - 2ydA = 0.$$

[1] Of course it depends on your parametrization.

8.1.25 Sketch the proof of Green's Theorem for the region shown in Figure 8.1.10.

It's enough here to split horizontally at the "local maximum"of the bottom of the region.

8.2.8 Let C be the closed piecewise smooth curve formed by traveling in straight lines between the points $(0,0,0),(2,1,5),(1,1,3)$, and back to the origin, in that order. Use Stokes' Theorem to evaluate the integral:

$$\int_C xyzdx + xydy + xdz.$$

Three points define a plane. In this case, these three points (connected by *straight lines*) lie in the plane $z = 2x + y$ (recall how to find the equation of a plane using the normal vector).

The curl of $\mathbf{F} = (xyz, xy, x)$ equals to $\nabla \times \mathbf{F} = (0, xy - 1, y - xz)$.

Parametrizing the plane $z = 2x + y = f(x,y)$ as a function, $\mathbf{\Phi}(x,y) = (x, y, f(x,y))$, which has upward facing normal $\mathbf{\Phi}_x \times \mathbf{\Phi}_y = (-f_x, -f_y, 1) = (-2,-1,1)$. We need this normal vector, because looking at the curve from above, we see that in the $xy-$plane it travels counterclockwise: $(0,0) \to (2,1) \to (1,1) \to (0,0)$. Therefore Stokes' Theorem tells us we need the upward facing normal.

Now we need to compute

$$\nabla \times \mathbf{F} \cdot d\mathbf{S} = -(xy-1) + (y - x(2x+y))dxdy = 1 - xy + y - 2x^2 - xydxdy$$
$$= 1 + y - 2xy - 2x^2dxdy.$$

By Stokes' Theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_D \nabla \times \mathbf{F} dA = \int_0^1 \int_y^{2y} 1 + y - 2xy - 2x^2 dxdy$$
$$= \int_0^1 y + y^2 - \frac{23}{3}y^3 dy = \frac{1}{2} + \frac{1}{3} - \frac{23}{12} = -\frac{13}{12}.$$

If you just computed the line integral, you would also have to integrate a cubic polynomial.

8.2.14 Let $\mathbf{c}$ consist of straight lines joining $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, and let S be the triangle with these vertices. Verify Stokes' Theorem directly with $\mathbf{F} = (yz, xz, xy)$.

We've done the line integral part of a very similar question (see midterm 2 solutions). I will only do the double integral here: to set up the region (same as the above question), consider the triangle made in the $xy-$plane: it travels counterclockwise $(1,0) \to (0,1) \to (0,0) \to (1,0)$. So we need the upward normal vector. In this case, all lines are in the plane $z = 1 - x - y$. The upward normal from the usual function parametrization of a surface is $(1, 1, 1)$. The curl of $\mathbf{F}$ turns out to be zero, so the integral is just zero.

Notice that $\mathbf{F} = \nabla(xyz)$, so the Fundamental Theorem of Line Integrals also tells us the path integral is zero.

8.2.18 Find $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$, where S is the ellipsoid $x^2 + y^2 + 2z^2 = 10$ and $\mathbf{F}$ is the vector field $(\sin(x, y), e^x, -yz)$.

This looks like a nice problem for going the other way with Stokes' Theorem: there is no boundary, so we can split up S into two regions with the same boundary, with the boundary curve of one having opposite orientation of the other. By Stokes' Theorem,

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \nabla \times \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{s} + \int_{-C} \mathbf{F} \cdot d\mathbf{s}.$$

You should justify why the curves have opposite orientation (similar to the argument for Green's Theorem for general regions, e.g., 8.1.25 above).

8.2.23 Consider two surfaces S_1 and S_2 with the same boundary C. Describe with sketches how S_1 and S_2 must be oriented to ensure

$$\int_{S_1} \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_{S_2} \nabla \times \mathbf{F} \cdot d\mathbf{S}.$$

By considering the special case $S_1 = S_2$, we see that they need to be oriented opposite with respect to each other in order for these two integrals to be equal.

8.3.1 Determine which of the following vector fields $\mathbf{F}$ in the plane is the gradient of a scalar function f. If such an f exists, find it.

(a) $\mathbf{F} = (x, y)$ is the gradient of $(x^2 + y^2)/2$.

(b) $\mathbf{F} = (xy, xy)$ is not a gradient since its curl is $(0, 0, y - x)$.

(c) $\mathbf{F} = (x^2 + y^2, 2xy)$ has zero curl, and so I found that it's the gradient of $x^3/3 + xy^2$.

8.3.17 Determine if the following vector fields $\mathbf{F}$ are gradient fields. If there exists a function f such that $\nabla f = \mathbf{F}$, find f.

(a) $\mathbf{F} = (2xyz, x^2z, x^2y)$: by observation, it's easy to tell $\mathbf{F} = \nabla x^2yz$.

(b) $\mathbf{F} = (x\cos(y), x\sin(y))$ has nonzero curl.

(c) $\mathbf{F} = (x^2e^y, xyz, e^z)$ has nonzero curl since the $\mathbf{i}$ component is $-xy$.

(d) $\mathbf{F} = (2x\cos(y), -x^2\sin(y))$ is the gradient of $x^2\cos(y)$.

8.3.28 Let $\mathbf{G}$ be the vector field on $\mathbb{R}^3 - \{(x,y,z) : x = y = 0\}$ defined by

$$\mathbf{G} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right).$$

(a) Show that $\mathbf{G}$ is irrotational.

This is because the curl is zero.

(b) Show that flow lines of $\mathbf{G}$ are circles.

Check that $\mathbf{F}(R\cos(t), R\sin(t)) = (R\cos(t), R\sin(t))'$.

(c) How can we resolve the fact that the trajectories of $\mathbf{F} = (-y, x)$ and $\mathbf{G}$ are both the same, yet $\mathbf{F}$ is rotational and $\mathbf{G}$ is not?

Being rotational is a local (infinitesimal) property that has nothing to do with flow lines. G rotates something in a circle while never rotating it locally, like moving in a circle while always facing forward, while F rotates in a circle while rotating locally, like driving around in a circle.

8.4.12 Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = (3xy^2, 3x^2y, z^3)$ and S is the surface of the unit sphere (centered at the origin).

Use the Divergence Theorem:

$$\oiint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \nabla \cdot \mathbf{F} dV.$$

The divergence of $\mathbf{F}$ turns out to be $3(x^2+y^2+z^2) = 3\rho^2$. Changing variables to spherical, the integral becomes

$$\int_0^{2\pi} \int_0^{\pi} \int_0^1 3\rho^4 \sin(\phi) d\rho d\phi d\theta = 2\pi \cdot 2 \cdot \frac{3}{5} = \frac{12\pi}{5}.$$

8.4.14 Let W be the three-dimensional solid enclosed by the surfaces $x = y^2, x = 9, z = 0$, and $x = z$. Let S be the boundary of W. Use the Divergence Theorem to find the flux of $\mathbf{F}(x,y,z) = (3x - 5y, 4z - 2y, 8yz)$ across S.

Since the divergence of the vector field is just $1 + 8y$, the hard part is setting up the triple integral. I know that $y^2 \le x \le 9$ because $y^2 \le 9$ is bounded and $9 \le y^2$ is not, and I also know $0 \le z \le x$ since x is positive. To get the $y-$bounds, I have no additional information, and therefore I must set the $x-$bounds equal: $y^2 = 9$ so $-3 \le y \le 3$. The triple integral is

$$\int_{-3}^{3} \int_{y^2}^{9} \int_0^x 1 + 8y dz dx dy = \int_{-3}^{3} \int_{y^2}^{9} (1+8y) x dx dy = \int_{-3}^{3} (1+8y) \left(\frac{81 - y^4}{2} \right) dy.$$

If we factor out the left term, notice that $8y$(even function) is an odd function over the interval $[-3, 3]$, so its integral is zero. We are left with

$$= \int_{-3}^{3} \left(\frac{81 - y^4}{2} \right) dy = 243 - 243/5 = 972/5.$$

8.4.15 Evaluate $\iint_{\partial W} \mathbf{F} \cdot \mathbf{n} dA$, where $\mathbf{F}(x, y, z) = (x, y, -z)$ and W is the unit cube in the first octant. Perform the calculation directly and check by using the Divergence Theorem.

I partially did this problem in class, but accidentally used a different cube. The notation above is the same as $\mathbf{F} \cdot d\mathbf{S}$. We break up this surface integral into six pieces: the sides $x = 0, 1; y = 0, 1; z = 0, 1$.

On either one of $x, y, z = 0$, $\mathbf{F}$ will have a zero component in the same place that the normal vector has its only nonzero component.

- On $x = 1$, the normal is $\mathbf{n} = (1, 0, 0)$ and $\mathbf{F} = (x, y, -z) = (1, 0, 0)$, so the dot product is just 1.
- On $y = 1$, the normal is $\mathbf{n} = (0, 1, 0)$ and $\mathbf{F} = (x, y, -z) = (0, 1, 0)$, so the dot product is just 1.
- On $z = 1$, the normal is $\mathbf{n} = (0, 0, 1)$ and $\mathbf{F} = (x, y, -z) = (0, 0, -1)$, so the dot product is -1.

Summing over all six sides, we get $0 + 0 + 0 + 1 + 1 + -1 = 1$.

Using the Divergence Theorem, $\nabla \cdot \mathbf{F} = 1 + 1 - 1 = 1$ as well, and we are integrating over the unit cube (the cube with side length 1 and therefore volume 1), so we get $\iiint_E 1 dV = 1$.

8.4.16 Evaluate the surface integral $\iint_{\partial W} \mathbf{F} \cdot \mathbf{n} dA$, where

$$\mathbf{F}(x, y, z) = (1, 1, z(x^2 + y^2)^2)$$

and W is the solid cylinder $x^2 + y^2 \leq 1, 0 \leq z \leq 1$.

Divergence Theorem, then changing to cylindrical, tells us

$$\iint_{\partial W} \mathbf{F} \cdot \mathbf{n} dA = \int_0^{2\pi} \int_0^1 \int_0^1 (r^2)^2 dz r dr d\theta = 2\pi \int_0^1 r^5 dr = \frac{\pi}{3}.$$

8.4.21 Prove Green's identities:

$$\iint_{\partial W} f \nabla g \cdot d\mathbf{S} = \iiint_W (f \nabla^2 g + \nabla f \cdot \nabla g) dV$$

and

$$\iint_{\partial W} (f \nabla g - g \nabla f) \cdot d\mathbf{S} = \iiint_W (f \nabla^2 g - g \nabla^2 f) dV.$$

The second identity is an easy consequence of the first. The crux of the first identity is showing that $\nabla \cdot (f \nabla g) = f \nabla^2 g + \nabla f \cdot \nabla g$. To show this, you just need to know that $\nabla^2 g$ is notation for the Laplacian of g, $\nabla^2 g = \Delta g = g_{xx} + g_{yy} + g_{zz}$. Just take the divergence as you normally would.

8.5.1 Evaluate $\omega \wedge \eta$ if $\omega = xdx + ydy + zdz$ and $\eta = zdx \wedge dy + xdy \wedge dz + ydz \wedge dx$.

The answer is $(x^2 + y^2 + z^2)dx \wedge dy \wedge dz$.

8.5.2 Prove the following, which shows dot product is also a special case of a wedge product:

$$(a_1dx + a_2dy + a_3dz) \wedge (b_1dy \wedge dz + b_2dz \wedge dx + b_3dx \wedge dy) = \left(\sum_{i=1}^{3} a_ib_i\right) dx \wedge dy \wedge dz.$$

8.5.3 Find d of $\omega = xdx \wedge dy + zdy \wedge dz + ydz \wedge dx$.

This is just correctly using the distribution and anticommutativity relations for wedge product.

8.5.11 Let T be the triangular solid bounded by $x = 0, y = 0, z = 0$, and the plane $2x + 3y + 6z = 12$. Compute

$$\iint_{\partial T} F_1dxdy + F_2dzdx + F_3dxdy$$

directly and by Stokes' Theorem, if

(a) $F_1 = 3y, F_2 = -12, F_3 = 18z$; and

(b) $F_1 = z, F_2 = y, F_3 = x^2$.

By the definition of integrating a $2-$form from page 481,

$$\iint_S F_1dydz + F_2dzdx + F_3dxdy = \iint_D \left(F_1(\mathbf{\Phi})\frac{\partial(y,z)}{\partial(u,v)} + F_2(\mathbf{\Phi})\frac{\partial(z,x)}{\partial(u,v)} + F_3(\mathbf{\Phi})\frac{\partial(x,y)}{\partial(u,v)} \right) dudv,$$

where

$$\frac{\partial(a,b)}{\partial(u,v)} = \begin{vmatrix} \partial a/\partial u & \partial a/\partial v \\ \partial b/\partial u & \partial b/\partial v \end{vmatrix}.$$

With this notation, the general Stokes' Theorem simply says

$$\iint_S F_1dydz + F_2dzdx + F_3dxdy = \iiint_E \nabla \cdot (F_1, F_2, F_3)dV,$$

which you can compute yourself. I will just simplify the surface integrals to double integrals below.

First, let's parametrize the plane by $\mathbf{\Phi}(x,y) = (x, y, 2 - y/2 - x/3)$. Then we can compute for $z = 2 - y/2 - x/3$ that:

$$\frac{\partial(x,y)}{\partial(x,y)} = 1, \frac{\partial(y,z)}{\partial(x,y)} = 1/3, \frac{\partial(z,x)}{\partial(x,y)} = 1/2.$$

So the integral above is just equal to

$$\iint_D (F_1/3 + F_2/2 + F_3)dA = \int_0^6 \int_0^{4-2x/3} (F_1/3 + F_2/2 + F_3)dydx.$$

(a)

$$\int_0^6 \int_0^{4-2x/3} (F_1/3+F_2/2+F_3)dydx = \int_0^6 \int_0^{4-2x/3} (y-6+18(2-y/2-x/3))dydx.$$

(b)

$$\int_0^6 \int_0^{4-2x/3} (F_1/3+F_2/2+F_3)dydx = \int_0^6 \int_0^{4-2x/3} ((2-y/2-x/3)/3+y/2+x^2)dydx.$$

Let me know if there are any other problems from chapter 8 you weren't sure about.