Homework 11 Solutions

7.6.2 Evaluate the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

where $\mathbf{F}(x, y, z) = (x, y, z^2)$ and S is the surface parametrized by $\mathbf{\Phi}(u, v) = (2\sin(u), 3\cos(u), v)$ with $0 \le u \le 2\pi$ and $0 \le v \le 1$.

The solution below assumes the surface is parametrized outwards.

First,

$$\mathbf{F}(\mathbf{\Phi}(u, v)) = (2\sin(u), 3\cos(u), v^2).$$

Next, we have to find the outward normal vector:

$$\mathbf{\Phi}_u \times \mathbf{\Phi}_v = (2\cos(u), -3\sin(u), 0) \times (0, 0, 1) = (-3\sin(u), -2\cos(u), 0).$$

This is the inward facing vector, so we should instead take its negative:

$$d\mathbf{S} = (3\sin(u), 2\cos(u), 0)dudv.$$

So

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^{2\pi} 6\sin^2(u) + 6\cos^2(u)dudv = 12\pi.$$

7.6.4 Let $\mathbf{F}(x, y, z) = (2x, -2y, z^2)$. Evaluate

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

where S is the cylinder $x^2 + y^2 = 4$ with $z \in [0, 1]$.

Note that the top and bottom are not explicitly included in this integral. If it said the boundary of the solid cylinder, they would be.

So we just need to integrate the cylinder, which we can parametrize by $\mathbf{\Phi}(z, \theta) = (2\cos(\theta), 2\sin(\theta), z)$, which (you can check) has outward normal

$$\mathbf{\Phi}_\theta \times \mathbf{\Phi}_z = (2\cos(\theta), 2\sin(\theta), 0).$$

So

$$\begin{aligned}\mathbf{F} \cdot d\mathbf{S} &= (2\cos(\theta), -2\sin(\theta), z^2) \cdot (2\cos(\theta), 2\sin(\theta), 0)d\theta dz \\ &= 4\cos^2(\theta) - 4\sin^2(\theta)d\theta dz = 4\cos(2\theta)d\theta dz.\end{aligned}$$

Integrating this over $0 \le \theta \le 2\pi$ and $0 \le z \le 1$ gives 0 because of the integral of $\cos(2\theta)$.

7.6.12 A restaurant is being built on the side of a mountain. The architect's plans are shown in Figure 7.6.11 (it's the surface with side the cylinder $x^2 + (y - R)^2 = R^2$, bounded below by the surface $x^2 + y^2 + z = 4R^2 = (2R)^2$, and top $z = 4R^2$).

(a) The vertical curved wall of the restaurant is to be built of glass. What will be the surface area of this wall?

To parametrize the cylinder $x^2+(y-R)^2 = R^2$, we can let $x = R\cos(\theta)/2$, $y - R = R\sin(\theta)$ and $z = z$ for $0 \leq \theta \leq 2\pi$ and $4R^2 - (R\cos(\theta))^2 - (R + R\sin(\theta))^2 \leq z \leq 4R^2$. These are the theta bounds because our parametrization gives circles centered at $(0, R)$ for constant radii, where $\theta = 0$ is perpendicular to the $y-$axis. Check that

$$\mathbf{\Phi}_\theta \times \mathbf{\Phi}_z(\theta, z) = (R\cos(\theta), R\sin(\theta), 0), \tag{1}$$

and therefore

$$\|\mathbf{\Phi}_\theta \times \mathbf{\Phi}_z\| = R.$$

so, after simplifying the bounds for z, the integral is

$$\int_0^{2\pi}\int_{2R^2(1-\sin(\theta))}^{4R^2} Rdzd\theta = \int_0^{2\pi} 2R^3 + 2R^3\sin(\theta)d\theta = 4\pi R^3.$$

(b) To be large enough to be profitable, the consulting engineer informs the developer that the volume of the interior must exceed $\pi R^4/2$. For what R does the proposed structure satisfy this requirement?

Let's use the cylindrical coordinates from part (a) so that $x = r\cos(\theta), y = R + r\sin(\theta)$ and $z = z$, where $0 \leq \theta \leq 2\pi$, $0 \leq r \leq R$, and z is bounded between $4R^2-(r\cos(\theta))^2-(R+r\sin(\theta))^2 = 3R^2-r^2-2rR\sin(\theta)$ and $4R^2$. The Jacobian is the same as regular cylindrical because adding constants to x or y in a change of variables doesn't change the Jacobian. The volume integral is

$$\int_0^{2\pi}\int_0^R\int_{3R^2-r^2-2rR\sin(\theta)}^{4R^2} rdzdrd\theta = \int_0^{2\pi}\int_0^R [R^2+r^2+2rR\sin(\theta)]rdrd\theta$$

$$= \int_0^{2\pi} \frac{R^4}{2} + \frac{R^4}{4} + \frac{2R^4}{3}\sin(\theta)d\theta = \frac{3\pi R^4}{2}.$$

So for all R the volume exceeds the requirement. Notice that the volume must be less than half the volume of the cylinder, which is $2\pi R^4$. However, it might be surprising that it's a constant function of R^4.

(c) During a typical summer day, the environs of the restaurant are subject to a temperature given by

$$T(x, y, z) = 3x^2 + (y - R)^2 + 16z^2.$$

A heat flux density $-k\nabla T$ (k is a constant depending on the insulation to be used) through all sides of the restaurant (including the top and the contact with the hill) produces a heat flux. What is this total heat flux? (Your answer will depend on R and k.)

The gradient of T is $(6x, 2(y-R), 32z)$.

We split the surface integral into three parts, as in part (a). We start with the cylinder. In cylindrical coordinates (from the past two parts) the gradient of T is equal to $(6R\cos(\theta), 2R\sin(\theta), 32z)$.

From equation (1), part (a), $d\mathbf{S} = (R\cos(\theta), R\sin(\theta), 0)dzd\theta$. However, we need the unit vector to point inward because we are thinking about temperature coming into the restaurant. So we reverse the sign and get

$$-k\nabla T\cdot(-R\cos(\theta), -R\sin(\theta), 0) = k(6R^2\cos^2(\theta)+2R^2\sin^2(\theta)) = k(4R^2\cos^2(\theta)+2).$$

So the flux through this part is

$$\int_0^{2\pi}\int_{2R^2(1-\sin(\theta))}^{4R^2} k(4R^2\cos^2(\theta)+2)dzd\theta$$

$$= \int_0^{2\pi} k(4R^2\cos^2(\theta)+2)(2R^2+2R^2\sin(\theta))d\theta = 16\pi kR^4.$$

Next, the top has inward normal vector $(0,0,-1)$ in the region $x^2+(y-R)^2 = R^2$. In this case $-k\nabla T = -k(0,0,32)$ and the integral becomes

$$\iint_{x^2+(y-R)^2\leq R^2} 32kdA = 32\pi kR^2$$

The bottom part of the restaurant can be parametrized by $(x, y, 4R^2-x^2-y^2)$ and has upward normal $(2x, 2y, 1)$. In this case

$$-k\nabla T\cdot(2x,2y,1) = -k(12x^2+4y(y-R)+32(4R^2-x^2-y^2))$$

$$= k(20x^2+28y^2+4Ry-128R^2).$$

The integral is

$$\iint_{x^2+(y-R)^2\leq R^2} k(20x^2+28y^2+4Ry-128R^2)dA.$$

In the usual polar coordinates, $x = r\cos(\theta), y = r\sin(\theta)$, the region becomes $r^2 \leq 2rR\sin(\theta)$. Setting these two equal, $r = 0$ or $r = 2R\sin(\theta)$. Now setting these limits equal, $\theta = 0, \pi$. The integral becomes

$$\int_0^{\pi}\int_0^{2R\sin(\theta)} k(20r^2+8r^2\sin^2(\theta)+4Rr\sin(\theta)-128R^2)rdrd\theta$$

$$= k\int_0^{\pi}\int_0^{2R\sin(\theta)} 20r^3+8r^3\sin^2(\theta)+4Rr^2\sin(\theta)-128R^2rdrd\theta$$

$$= k\int_0^{\pi} 5(2R\sin(\theta))^4+2(2R\sin(\theta))^4\sin^2(\theta)+\frac{4R}{3}(2R\sin(\theta))^3\sin(\theta)-64R^2(2R\sin(\theta))^2d\theta.$$

We use the following:

$$\int_0^{\pi} \sin^{2n}(\theta) d\theta = \frac{1}{4^n}\binom{2n}{n}\pi.$$

We need it for $n = 1, 2, 3$. (You are expected to compute these by hand for these values of n...) Using this, the above integral equals

$$k(30\pi R^4 + 10\pi R^4 + 4\pi R^4 - 128\pi R^4) = -84\pi R^4.$$

So the total flux is $kR^2(32\pi - 68\pi R^2)$.

(Please let me know if you find a typo anywhere.)

7.6.15 Let S be the surface of the unit sphere. Let $\mathbf{F}$ be a vector field and F_r its radial component. Prove that

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^{\pi} F_r \sin(\phi) d\phi d\theta.$$

What is the corresponding formula for real-valued functions f?

By definition of radial part, $F_r = \mathbf{F} \cdot \mathbf{n}$ where $\mathbf{n}$ is the unit normal of the sphere (it's a function of θ, ϕ but not ρ, hence it's radial: only depends on a fixed radius). Using this same notation for $\mathbf{n}$, we have computed before that for the unit sphere, $d\mathbf{S} = \mathbf{n} \sin(\phi) d\phi d\theta$. Using this definition, the formula is clear since $\mathbf{n} \cdot \mathbf{n} = 1$.

I'm not sure what they mean by a corresponding formula for functions f. According to the back of the book, they just want you to compute what dS is ...? The only radial part I can think of for a real-valued function is $|f|$.

7.6.17 Work out a formula like that in Exercise 15 for integration over the surface of a cylinder.

Now the radial part of $\mathbf{F}$ is such that $F_r = \mathbf{F} \cdot \mathbf{n}$ where $\mathbf{n} = (\cos(\theta), \sin(\theta), 0)$, the unit normal of a cylinder. Similarly to the above problem, we have computed (e.g., problem 7.6.12) $d\mathbf{S} = (R\cos(\theta), R\sin(\theta), 0) dz d\theta = \mathbf{n} R dz d\theta$. So in this case, integrating over a cylinder of radius R, the formula is

$$\int_0^{2\pi} \int_a^b F_r R dz d\theta = R \int_0^{2\pi} \int_a^b F_r dz d\theta.$$

7.6.21 For $a, b, c > 0$, let S be the upper half ellipsoid

$$S = \left\{ (x, y, z) \in \mathbf{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, z \geq 0 \right\},$$

with orientation determined by the upward normal. Compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = (x^3, 0, 0)$.

I assume they don't mean the half ellipsoid closed surface (including the bottom), as we have already done surface integrals over functions z =constant many times. We can parametrize a surface contained in the above ellipsoid by $\mathbf{\Phi}(\phi, \theta) = (a\cos(\theta)\sin(\phi), b\sin(\theta)\sin(\phi), c\cos(\phi))$. Then the outward normal vector with respect to $\mathbf{\Phi}$ is

$$\mathbf{\Phi}_\phi \times \mathbf{\Phi}_\theta = \sin(\phi)(bc\cos(\theta)\sin(\phi), ac\sin(\theta)\sin(\phi), ab\cos(\phi)).$$

By definition of $\mathbf{F}$, you only needed to compute the first component, since that's all the dot product sees. Now

$$\mathbf{F}\cdot d\mathbf{S} = bc\sin(\phi)(a\cos(\theta)\sin(\phi))^3(\cos(\theta)\sin(\phi))d\phi d\theta = a^3bc\cos^4(\theta)\sin^5(\phi)d\phi d\theta.$$

Finally,

$$\int_S \mathbf{F}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\pi} a^3bc\cos^4(\theta)\sin^5(\phi)d\phi d\theta = a^3bc\left(\int_0^{2\pi}\cos^4(\theta)d\theta\right)\left(\int_0^{\pi}\sin^5(\phi)d\phi\right).$$

The theta integral requires two double angle formulas and the phi integral requires the trig sub $\sin^4(\theta) = (1-\cos^2(\theta))^2$; you can plug into Wolfram Alpha to get the answer.

7.7.8 Find the curvature K of the following.

(a) The cylinder $\mathbf{\Phi}(u, v) = (2\cos(v), 2\sin(v), u)$.

We compute:

$$\mathbf{\Phi}_u = (0,0,1), \qquad \mathbf{\Phi}_v = (-2\sin(v), 2\cos(v), 0),$$
$$\mathbf{\Phi}_{uu} = (0,0,0), \mathbf{\Phi}_{uv} = (0,0,0), \qquad \mathbf{\Phi}_{vv} = (-\cos(v), -\sin(v), 0).$$

Now it doesn't matter what $\mathbf{N}$ is because $\ell = m = 0$. Therefore K is always zero. This might seem weird, but K is actually the product of principal curvatures: the curvature perpendicular to the $xy-$plane is always zero, although the curvature parallel to the $xy-$plane is not zero. It turns out H is the average of these two sectional curvatures, and in this case it would not be zero, but the radius of the cylinder divided by two. (You can compute to see that these two curvatures multiply to zero and sum to $1/2$, and therefore one is zero and the other is $1/2$. But $1/2$ is exactly the curvature of a circle of radius 2. You can think of the curvature $1/R^2$ of a sphere as the product of its two principal curvatures $1/R$ of the circles contained in it.)

(b) The surface $\mathbf{\Phi}(u, v) = (u, v, u^2)$.

Again we first compute

$$\mathbf{\Phi}_u = (1,0,2u), \qquad \mathbf{\Phi}_v = (0,1,0),$$
$$\mathbf{\Phi}_{uu} = (0,0,2), \qquad \mathbf{\Phi}_{uv} = (0,0,0), \qquad \mathbf{\Phi}_{vv} = (0,0,0).$$

Now $m = n = 0$, so again $K = 0$. This is because we just have $z = x^2$ and cross sections along y =constant are just lines, so there is a direction in which a principal curvature vanishes.

7.7.9 Show that Enneper's surface

$$\mathbf{\Phi}(u,v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + u^2v, u^2 - v^2\right)$$

is a minimal surface ($H = 0$).

Again we compute

$$\mathbf{\Phi}_u = (1 - u^2 + v^2, 2uv, 2u), \quad \mathbf{\Phi}_v = (2uv, 1 - v^2 + u^2, -2v),$$
$$\mathbf{\Phi}_{uu} = (-2u, 2v, 2), \mathbf{\Phi}_{uv} = (2v, 2u, 0), \qquad \mathbf{\Phi}_{vv} = (2u, -2v, -2).$$

Now you can check that these equations were made so that

$$E = G = (1 + u^2 + v^2)^2 \text{ and } F = 0.$$

Also, we need $\mathbf{\Phi}_u \times \mathbf{\Phi}_v = (-2u(1 + u^2 + v^2), 2v(1 + u^2 + v^2), 1 - (u^2 + v^2)^2)$, since

$$\mathbf{N} = \frac{\mathbf{\Phi}_u \times \mathbf{\Phi}_v}{\sqrt{EG - F^2}} = \left(-2u, 2v, \frac{1 - (u^2 + v^2)^2}{1 + u^2 + v^2}\right).$$

Now

$$G\ell = G\mathbf{N} \cdot \mathbf{\Phi}_{uu} = (1 + u^2 + v^2)(4(u^2 + v^2)(1 + u^2 + v^2) + 2(1 - (u^2 + v^2)^2))$$

and you can compute that

$$En = -G\ell.$$

Since $F = 0$, the mean cuvature is $H = (G\ell + En - 2Fm)/(2W) = 0/(2W) = 0$. Therefore this surface is minimal.