1. (10 points) Define the curve $\mathbf{c}(t) = (t, \ln(t), e^{t^2})$ in the interval $1 \leq t \leq 2$. Let

$$\mathbf{F} = \big(ye^{xy}\sin(xz) + e^{xy}z\cos(xz), xe^{xy}\sin(xz), xe^{xy}\cos(xz)\big).$$

Compute $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$.

Solution: Notice that $\mathbf{F} = \nabla f$ where $f = e^{xy}\sin(xz)$. By the fundamental theorem of line integrals,

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = f(\mathbf{c}(2)) - f(\mathbf{c}(1)) = f(2, \ln(2), e^4) - f(1, 0, e) = 4\sin(2e^4) - \sin(e).$$

2. (10 points) Consider the piece of the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

in the first octant.

(a) Where does the plane intersect the x, y, and z axes?

(b) Find the surface area of this piece of the plane.

(c) Consider the tetrahedron made by this plane and the triangles it makes in the xy, yz, and xz planes (the coordinate planes). Find the areas of the remaining three triangles.

(d) (Extra credit:) Find a formula relating the areas of the three triangles in the coordinate planes and the area of the "hypotenuse"plane from part (a).

Solutions:

(a) Plugging in $y = 0 = z$ gives us the $x-$intercept $x = a$. Similarly, $y = b$ and $z = c$ and the y, z intercepts, respectively. Note that this is a convenient way to write the equation of a plane in "intercept-form."

(b) The surface area of a function is given by

$$\iint_S \sqrt{1 + f_x^2 + f_y^2} dA.$$

In this case, we can solve for $z = c - cx/a - cy/b$ and set up the integral as

$$\int_0^b \int_0^{a-ay/b} \sqrt{1 + c^2/a^2 + c^2/b^2} dxdy = \sqrt{1 + c^2/a^2 + c^2/b^2} \int_0^b a - ay/bdy$$

$$= \sqrt{1 + c^2/a^2 + c^2/b^2}(ab - ab/2) = \frac{1}{2}\sqrt{a^2b^2 + b^2c^2 + a^2c^2}.$$

(c) The triangles are right triangles with side lengths found in (a). The respective areas are $ab/2$, $bc/2$ and $ac/2$.

(d) Letting $A = bc/2$, $B = ac/2$, $C = ab/2$, and $D = \frac{1}{2}\sqrt{a^2b^2 + b^2c^2 + a^2c^2}$, notice the nice generalization of the Pythagorean Theorem for right tetrahedrons:

$$A^2 + B^2 + C^2 = D^2.$$

3. (10 points) Let $\mathbf{F} = (x, -2y, z)$ be a vector field on $\mathbb{R}^3$. Find the upward flux of the vector field through the surface given by $z = x^2y$ in the region $x^2 + 4y^2 = 1$, $x \geq 0$.

   Solution: The surface is the graph of a function and therefore can be parametrized by (x, y, xy), and has upward normal vector $(y, x, 1)$. The flux is given by the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{x^2+4y^2\leq 1} (x, -2y, x^2y) \cdot (2xy, x^2, 1)dA = \iint_{x^2+4y^2\leq 1} x^2y dA.$$

   We can change variables by $x = r\cos(\theta), y = r\sin(\theta)/2$, with $0 \leq r \leq 1, -\pi/2 \leq \theta \leq \pi/2$ in which case $dA = r/2$. Therefore the above integral becomes

$$\int_0^1 \int_{-\pi/2}^{\pi/2} r^3 \cos^2(\theta)\sin(\theta)/2(r/2)drd\theta = \frac{1}{4}\int_0^1 r^4 dr \int_{-\pi/2}^{\pi/2} \cos^2(\theta)\sin(\theta)d\theta = \frac{1}{4} \cdot \frac{1}{5} \cdot \left(-\frac{\cos^3(\theta)}{3}\Bigg|_{-\pi/2}^{\pi/2} \right) = 0.$$