1. (10 points) Let $D = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq 2, \quad 0 \leq y \leq x, \quad 0 \leq z \leq y\}$.
   Write $\iiint_D x^2 dV$ as an iterated integral in terms of $dzdydx$, $dzdxdy$, $dxdzdy$, and $dxdydz$.
   Solution: By switching two consecutive integrals at a time (draw the 2-D regions), we can see that

$$\begin{aligned}
&\iiint_D x^2 dV \\
&= \int_0^2 \int_0^x \int_0^y x^2 dzdydx \\
&= \int_0^2 \int_y^2 \int_0^y x^2 dzdxdy \\
&= \int_0^2 \int_0^y \int_y^2 x^2 dxdzdy \\
&= \int_0^2 \int_z^2 \int_y^2 x^2 dxdydz.
\end{aligned}$$

2. (5 points) Evaluate $\int_0^1 \int_{\sqrt{x}}^x e^{x/y} dy dx$.

Solution: Again, draw the 2-D region. Note that we are going from $y = \sqrt{x}$ to x, so the region is not oriented the way we learned for Fubini's Theorem. So we should first rewrite

$$\int_0^1 \int_{\sqrt{x}}^x e^{x/y} dy dx = -\int_0^1 \int_x^{\sqrt{x}} e^{x/y} dy dx.$$

Now we can draw the region or write the correct inequalities (this would have worked for Fubini the other way as well: $\sqrt{x} \leq y \leq x$ means $x \leq y^2 \leq x^2$ since all quantities are positive), and use Fubini's theorem to get

$$\int_0^1 \int_x^{\sqrt{x}} e^{x/y} dy dx = \int_0^1 \int_{y^2}^y e^{x/y} dx dy = \int_0^1 y e^{x/y}|_{y^2}^y dy = \int_0^1 ye - ye^y dy = y^2(e/2) - (ye^y - e^y)|_0^1 = e/2 - 1.$$

So the final answer is $-e/2 + 1$.

3. (5 points) Evaluate $\int_0^\pi \int_0^1 \int_0^1 e^{z^2 y^2 \sin(yz)} \cos(x) dz dy dx$.

Solution: Since all limits of integration are scalars, and the function we are integrating can be written as $f(x, y, z) = g(x)h(y, z)$, we can split the integral into

$$\int_0^\pi \cos(x) dx \int_0^1 \int_0^1 e^{z^2 y^2 \sin(yz)} dz dy.$$

The left integral is zero, so the final answer is zero.

4. (10 points) Bound the integral $\int_0^{10}\int_0^{10}\int_0^{10} \sin(x^3y^3z^3)dxdydz$ above and below. What is your opinion of the bounds?

Solution: Since $-1 \leq \sin(x) \leq 1$ for *any* x, we know

$$\int_0^{10}\int_0^{10}\int_0^{10} -1dxdydz \leq \int_0^{10}\int_0^{10}\int_0^{10} \sin(x^3y^3z^3)dxdydz \leq \int_0^{10}\int_0^{10}\int_0^{10} 1dxdydz.$$

The leftmost integral is equal to -10^3 and the rightmost integral is 10^3. So

$$-10^3 \leq \int_0^{10}\int_0^{10}\int_0^{10} \sin(x^3y^3z^3)dxdydz \leq 10^3.$$

However, this is a bad approximation since $\sin(x^3y^3z^3)$ oscillates a lot in any large box. (Unfortunately, basic WolframAlpha times out during the computation of the above integral.)

5. (6 points) Extra Credit: only using triple integrals and changes of variables[1], show that the volume of the ellipsoid

$$E = \{(x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1\}$$

is equal to abc times the volume of the unit ball $B = \{(u, v, w) \in \mathbb{R}^3 : u^2 + v^2 + w^2 \le 1\}$.

Solution: The quick solution (but you would need to be very comfortable with change of variables) is let $x = au, y = bv, z = cw$. Then the region $\{x^2/a^2 + y^2/b^2 + z^2/c^2 \le 1\}$ changes to the region $\{u^2 + v^2 + w^2 \le 1\}$. The volume of the new region is just $4\pi/3$. Now, if $x = au$, thinking of the integral as iterated integrals, $dx = adu$, and so on. Therefore

$$\iiint_E dxdydz = \iiint_B (adu)(bdv)(cdw) = abc \iiint_B dudvdw = abc\text{Vol}(B) = abc4\pi/3.$$

A slightly longer way that you might be more comfortable with would be to write the limits explicitly and change one variable at a time (obeying the Calculus 2 rules):

$$\iiint_E dxdydz = \int_{-c}^{c} \int_{-b\sqrt{1-z^2/c^2}}^{b\sqrt{1-z^2/c^2}} \int_{-a\sqrt{1-y^2/b^2-z^2/c^2}}^{a\sqrt{1-y^2/b^2-z^2/c^2}} dxdydz.$$

Now let $x = au$ so that $adu = dx$ and the limits change by $u = x/a$. We get the above integral equals

$$\int_{-c}^{c} \int_{-b\sqrt{1-z^2/c^2}}^{b\sqrt{1-z^2/c^2}} \int_{-\sqrt{1-y^2/b^2-z^2/c^2}}^{\sqrt{1-y^2/b^2-z^2/c^2}} adudydz = \int_{-c}^{c} \int_{-b\sqrt{1-z^2/c^2}}^{b\sqrt{1-z^2/c^2}} 2a\sqrt{1 - y^2/b^2 - z^2/c^2}dydz.$$

Change variables again by $y = bv$ to get the above equals

$$\int_{-c}^{c} \int_{-\sqrt{1-z^2/c^2}}^{\sqrt{1-z^2/c^2}} 2ab\sqrt{1 - v^2 - z^2/c^2}dvdz.$$

Now let $f(z) = \int_{-b\sqrt{1-z^2/c^2}}^{b\sqrt{1-z^2/c^2}} 2ab\sqrt{1 - v^2 - z^2/c^2}dv$. The last remaining integral is

$$\int_{-c}^{c} f(z)dz,$$

which is just a single integral and so it obeys all the change of variables you previously learned. Letting $z = cw$ finishes the problem, after you notice that the integral left is the same as abc times the volume of the unit ball.

[1] The change of variables you use should be from a previous class, not from Chapter 6. Remember that computing an iterated triple integral is just iterating a single integral three times, so all previous results hold for each integral separately.