1. (5 points) Let $f(x,y) = xe^{y-1} + y\ln x$.
   (a) Find $\nabla f(1,1)$.
   (b) Use the linearization (tangent plane) to approximate $.9e^{.2} + 1.2\ln(.9)$. (The actual value is $\approx .9728$.)

   Solution:

   (a) $\frac{\partial f}{\partial x} = e^{y-1} + y/x$. So $\frac{\partial f}{\partial x}(1,1) = 1 + 1 = 2$.
   $\frac{\partial f}{\partial y} = xe^{y-1} + \ln x$. So $\frac{\partial f}{\partial y}(1,1) = 1 + 0 = 1$.

   (b) $f(1,1) = 1$. The tangent plane is given by

   $$L(x,y) = f(1,1) + \frac{\partial f}{\partial x}(1,1)(x-1) + \frac{\partial f}{\partial y}(1,1)(y-1) = 1 + 2(x-1) + (y-1).$$

   To approximate,

   $$.9e^{.2} + 1.2\ln .9 = f(.9, 1.2) \approx L(.9, 1.2) = 1 + 2(.9-1) + (1.2-1) = 1 - .2 + .2 = 1,$$

   which is pretty close to .9728.

2. (5 points) Let $f(x,y)$ be as above. Use the chain rule to find $\nabla f(u,v)$ at $(u,v) = (1,1)$ if $x = u + v$ and $y = u - v$.

   Solution: The chain rule says

   $$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u},$$

   $$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

   If $(u,v) = (1,1)$, then $(x,y) = (u+v, u-v) = (2,0)$. We already computed the partial derivatives with respect to x, y above, so plug in the new point $(2,0)$ into ∇f to get

   $$\frac{\partial f}{\partial x} = e^{-1} + 0/2 = e^{-1}, \text{ and } \frac{\partial f}{\partial y} = 2e^{-1} + \ln 2.$$

   Moreover,

   $$\begin{bmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

   Therefore

   $$\frac{\partial f}{\partial u}(1,1) = e^{-1} + 2e^{-1} + \ln 2 = \frac{3}{e} + \ln 2,$$

   $$\frac{\partial f}{\partial v}(1,1) = e^{-1} - (2e^{-1} + \ln 2) = -\frac{1}{e} - \ln 2.$$

3. (10 points) (Maybe bonus) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. Assume there are two continuous paths $\mathbf{r}_1 : [-1, 1] \to \mathbb{R}^2$ and $\mathbf{r}_2 : [-1, 1] \to \mathbb{R}^2$ with $\mathbf{r}_1(0) = \mathbf{r}_2(0) = (0, 0)$ such that

$$\lim_{t\to 0} f(\mathbf{r}_1(t)) = A \text{ and } \lim_{t\to 0} f(\mathbf{r}_2(t)) = B,$$

where $A, B \in \mathbb{R}$.

(a) If f is continuous at $(0, 0)$, prove that $A = B$.

(b) Note that the above implies: if along two different paths there are two different limits, the limit $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist!

Use this to prove that $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ does not exist by considering the two paths $\mathbf{r}_1(t) = (t, t)$ and $\mathbf{r}_2(t) = (t^2, t)$ (this is the same as considering the limit along $x = y$ and along $x = y^2$ as they might in MTH 234).

Solutions:

(a) An argument without the limit definition, but using stuff we learned in class, could go as follows: if f is continuous at $(0, 0)$, and $\mathbf{r}_1$ is continuous, so is their composition $f(\mathbf{r}_1(t))$. Similarly, $f(\mathbf{r}_2(t))$ is continuous. By definition of continuity, since $\mathbf{r}_1(0) = \mathbf{r}_2(0) = (0, 0)$,

$$A = \lim_{t\to 0} f(\mathbf{r}_1(t)) \overset{\text{continuity of } \mathbf{r}_1}{=} f(0, 0) \overset{\text{continuity of } \mathbf{r}_2}{=} \lim_{t\to 0} f(\mathbf{r}_2(t)) = B.$$

An argument without using this fact would just prove that the composition of continuous functions is continuous:

Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $\mathbf{r} : [-1, 1] \to \mathbb{R}^2$ be continuous such that $\mathbf{r}(0) = (0, 0)$. Let $\varepsilon > 0$. We want to show there is $\delta > 0$ such that

$$|t - 0| < \delta \text{ implies } |f(\mathbf{r}(t)) - f(0, 0)| < \varepsilon.$$

Since f is continuous at $(0, 0)$, there is δ_1 such that

$$\|(x, y) - (0, 0)\| < \delta_1 \text{ implies } |f(x, y) - f(0, 0)| < \varepsilon.$$

Since $\mathbf{r}$ is continuous at $t = 0$, there is $\delta_2 > 0$ such that

$$|t - 0| < \delta_2 \text{ implies } \|\mathbf{r}(t) - (0, 0)\| < \delta_1.$$

Therefore, letting $\delta = \delta_2$, we see that

$$|t - 0| < \delta \text{ implies } |f(\mathbf{r}(t)) - f(0, 0)| < \varepsilon.$$

To finish this problem, since $f(\mathbf{r}(t))$ is continuous at $t = 0$, $A = f(0, 0) = B$.

(b) Along $\mathbf{r}_1$, the limit becomes

$$\lim_{t\to 0} \frac{t^3}{t^2 + t^4} = \lim_{t\to 0} \frac{t}{1 + t^2} = 0.$$

Along $\mathbf{r}_2$, the limit becomes

$$\lim_{t\to 0} \frac{t^4}{t^4 + t^4} = \lim_{t\to 0} \frac{t^4}{2t^4} = \frac{1}{2}.$$