1. (20 points) (Parts are unrelated to each other in this question.)

(a) Parametrize the line segment starting at $(1,1,1)$ and ending at $(2,1,3)$.

One parametrization is given by

$$\mathbf{r}(t) = (1,1,1)(1-t) + (2,1,3)t \text{ for } 0 \leq t \leq 1.$$

I didn't take off points if you forgot the domain of $\mathbf{r}$.

(b) Find the point where the line $(t, 1-t, 1-t)$ intersects the plane $2x - y - z = 0$.

Since $x(t) = t, y(t) = 1-t, z(t) = 1-t$, you should first plug these into the equation of the plane and solve for t:

$$2t - (1-t) - (1-t) = 0 \implies 4t = 2 \implies t = 1/2.$$

So the point is $(1/2, 1/2, 1/2)$.

(c) Given a C^1 function $f : \mathbb{R}^n \to \mathbb{R}$, what is the direction of greatest increase at a point $\mathbf{x}_0$?

The direction of greatest increase is the direction of the gradient. Since direction is a unit vector, the answer is $\nabla f(x_0)/\|\nabla f(x_0)\|$ if $\nabla f(x_0) \neq \mathbf{0}$. If it is the zero vector, then we might not be able to say what the direction of greatest increase is.

(d) Let $f(x,y) = ye^{\sin(y)} + xe^{\cos(x)}$. Find $\dfrac{\partial^4 f}{\partial x (\partial y)^3}(1,2)$.

By properties of differentiable functions (multiplication, composition, addition), $f(x,y)$ is infinitely differentiable. Therefore we can move around all the partials we want.

$$\frac{\partial^4 f}{\partial x (\partial y)^3} = \frac{\partial^4 f}{(\partial y)^3 \partial x} = \frac{\partial^3}{(\partial y)^3}(\text{some function of only } x) = 0,$$

since the partial derivative with respect to x of $ye^{\sin(y)}$ is the zero function.

2. (20 points)

Let $f(x, y) = \frac{x^3}{3} - x + \frac{y^2}{2} - y$. Let D be the closed region the triangle with vertices at $(0, 0)$, $(0, 2)$, and $(2, 2)$. Find the absolute maximum and minimum of f on D.

Solution: First, check where the gradient is zero inside the region:

$$\nabla f = (x^2 - 1, y - 1) = (0, 0) \text{ at } (\pm 1, 1).$$

The only point inside the region out of these two is $(1, 1)$.

Next, check the boundaries. The triangle has three sides: $y = x, 0 \leq x \leq 2$; $x = 0, 0 \leq y \leq 2$; and $y = 2, 0 \leq x \leq 2$.

- On $y = x$,
  $$f(x, x) = x^3/3 - x + x^2/2 - x = g_1(x).$$
  Then $g_1'(x) = x^2 + x - 2 = (x + 2)(x - 1)$. This gives us the point $(1, 1)$, which we already had above. $(-2, -2)$ is not inside the region.

- On $x = 0$,
  $$f(0, y) = y^2/2 - y = g_2(y).$$
  Then $g_2'(y) = y - 1 = 0$ when $y = 1$. This gives us the point $(0, 1)$.

- On $y = 2$,
  $$f(x, 2) = x^3/3 - x + 2 - 2 = x^3/3 - x = g_3(x).$$
  Then $g_3'(x) = x^2 - 1 = (x + 1)(x - 1)$, giving us the point $(1, 2)$.

Finally, we have three more points coming from the boundary of these lines, namely $(0, 0), (2, 0)$, and $(2, 2)$.

Plugging in each of these six points, the absolute minimum is $f(1, 1) = -7/6$ and the absolute maximum is $f(2, 2) = 2/3$.

3. (20 points) Let $f(x, y) = (e^x \sin(y), e^x \cos(y))$. If $x(t) = \cos(t)$ and $y(t) = \sin(t)$, *using the Chain Rule*, find $\dfrac{df}{dt}$ at $t = 0$.

Solution: Think of $\mathbf{c}(t) = (x(t), y(t))$ as a column vector. The chain rule tells us

$$\frac{df}{dt}(0) = Df(x(0), y(0))\mathbf{c}'(0).$$

First,

$$Df = \begin{bmatrix} \frac{\partial}{\partial x} e^x \sin(y) & \frac{\partial}{\partial y} e^x \sin(y) \\ \frac{\partial}{\partial x} e^x \cos(y) & \frac{\partial}{\partial y} e^x \cos(y) \end{bmatrix} = \begin{bmatrix} e^x \sin(y) & e^x \cos(y) \\ e^x \cos(y) & -e^x \sin(y) \end{bmatrix}.$$

Since $(x(0), y(0)) = (1, 0)$, Df evaluated at $t = 0$ is

$$Df(1, 0) = \begin{bmatrix} 0 & e \\ e & 0 \end{bmatrix}.$$

Also, $\mathbf{c}'(0) = (0, 1)^T$. Therefore,

$$\frac{df}{dt}(0) = \begin{bmatrix} 0 & e \\ e & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} e \\ 0 \end{bmatrix}.$$

4. (20 points)

(a) State the Implicit Function Theorem for C^1 functions $F : \mathbb{R}^{n+1} \to \mathbb{R}$.

See book.

(b) State the Inverse Function Theorem.

See book.

(c) Prove the Inverse Function Theorem using Implicit Function Theorem.

Solution: Assume that the system

$$\begin{cases} y_1 &= f_1(x_1, \ldots, x_n) \\ \vdots &= \quad \vdots \\ y_n &= f_n(x_1, \ldots, x_n) \end{cases}$$

is such that

$$J(f)(x_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_1}{\partial x_n} \\ \vdots \ddots \vdots \\ \frac{\partial f_n}{\partial x_1} \cdots \frac{\partial f_n}{\partial x_n} \end{bmatrix} (x_0)$$

is invertible. Let $F_i(y, x) = f_i(x) - y_i$ for $1 \leq i \leq n$. Then the function $F : \mathbb{R}^{2n} \to \mathbb{R}^n$ defined by $F(y, x) = (F_1(y, x), \ldots, F_n(y, x)) = 0$ at (y_0, x_0) (where $y_0 = (f_1(x_0), \ldots, f_n(x_0))$). Moreover, we have

$$D_x(F)(y_0, x_0) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} \cdots \frac{\partial F_1}{\partial x_n} \\ \vdots \ddots \vdots \\ \frac{\partial F_n}{\partial x_1} \cdots \frac{\partial F_n}{\partial x_n} \end{bmatrix} (y_0, x_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_1}{\partial x_n} \\ \vdots \ddots \vdots \\ \frac{\partial f_n}{\partial x_1} \cdots \frac{\partial f_n}{\partial x_n} \end{bmatrix} (x_0),$$

and therefore $D_x(F)(y_0, x_0)$ is invertible. So the (general) Implicit Function Theorem guarantees that $x_i = g_i(y_1, \ldots, y_n)$ in some open set containing (y_0, x_0) for $1 \leq i \leq n$ and g_i are continuously differentiable.

5. (20 points)

(a) A norm has 4 properties (including triangle inequality). Define what a norm on $\mathbb{R}^2$ is.

A norm on $\mathbb{R}^2$ is ***any*** function $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}$ satisfying

(i) $\|(x, y)\| \geq 0$ for all $(x, y) \in \mathbb{R}^2$;

(ii) $\|(x, y)\| = 0$ if and only if $(x, y) = (0, 0)$;

(iii) For $\alpha \in \mathbb{R}$, $\|\alpha(x, y)\| = |\alpha|\|(x, y)\|$;

(iv) $\|(x_1, y_1) + (x_2, y_2)\| \leq \|(x_1, y_1)\| + \|(x_2, y_2)\|$.

(b) Prove that the function $\|\cdot\|_1 : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\|(x, y)\|_1 = |x| + |y|$$

is a norm on $\mathbb{R}^2$. This is called the ℓ_1 norm on $\mathbb{R}^2$.

(i) $\|(x, y)\|_1 = |x| + |y| \geq 0$ for all $(x, y) \in \mathbb{R}^2$ since absolute value is always nonnegative;

(ii) $\|(x, y)\|_1 = 0$ if and only if $|x| + |y| = 0$, which only happens when $x = 0$ and $y = 0$ since absolute value is zero if and only if the input is zero;

(iii) For $\alpha \in \mathbb{R}$,

$$\|\alpha(x, y)\|_1 = \|(\alpha x, \alpha y)\|_1 = |\alpha x| + |\alpha y| = |\alpha||x| + |\alpha||y| = |\alpha|(|x| + |y|) = |\alpha|\|(x, y)\|_1;$$

(iv)

$$\begin{aligned} \|(x_1, y_1) + (x_2, y_2)\|_1 &= \|(x_1 + x_2, y_1 + y_2)\|_1 = |x_1 + x_2| + |y_1 + y_2| \\ &\leq |x_1| + |x_2| + |y_1| + |y_2| = (|x_1| + |y_1|) + (|x_2| + |y_2|) = \|(x_1, y_1)\|_1 + \|(x_2, y_2)\|_1. \end{aligned}$$

(c) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} x + y \text{ if } x \text{ or } y \text{ are rational,} \\ x - y \text{ if } x \text{ and } y \text{ are irrational.} \end{cases}$$

Prove that f is continuous at $(0, 0)$ with respect to the ℓ_1 norm (that is, in the definition of continuity, show that $\|(x, y) - (0, 0)\|_1 < \delta$ implies ...).

Note that

$$|x + y| \leq |x| + |y| = \|(x, y)\|_1 \text{ and } |x - y| \leq |x| + |y| = \|(x, y)\|_1.$$

Let $\varepsilon > 0$. Let $\delta = \varepsilon$. By the above computation, no matter what $(x, y) \in \mathbb{R}^2$ is,

$$|f(x, y) - f(0, 0)| \leq |x| + |y| = \|(x, y)\|_1.$$

Therefore, if $\|(x, y)\|_1 < \delta$ then $|f(x, y) - f(0, 0)| < \delta = \varepsilon$.

Bonus (10 points) Let $\mathbb{Q}$ be the set of rational numbers. Given a prime number p, any nonzero rational number x can be expressed as $x = p^a \cdot \frac{n}{d}$ for a unique integer a, where n and d are integers not divisible by p.

For a prime number p, define $|\cdot|_p : \mathbb{Q} \to \mathbb{Q}$ by $|x|_p = p^{-a}$, if x is as above, and define $|0|_p = 0$.

(a) Prove $|\cdot|_p$ is a norm on $\mathbb{Q}$ (is this even a well-defined function?).

This is well-defined because the expression for rational numbers given above is unique. As in the previous question, we check the properties of a norm:

(i), (ii) $|x|_p = p^{-a}$ where a is uniquely given if x is rational and nonzero. p^{-a} is strictly positive for any integer a. If $x = 0$, then by definition $|0|_p = 0$.

(iii) Since $|\cdot|_p$ is only defined for rationals, let $\alpha \in \mathbb{Q}$. α can be uniquely expressed as $p^b m/k$. Let $x = p^a n/d$ as above. Then

$$|\alpha x|_p = |p^{a+b} mn/(kd)|_p = p^{-a-b} = p^{-a}p^{-b} = |\alpha|_p |x|_p.$$

Above, it's easy to see that $mn/(kd)$ is such that the numerator and denominator both don't have any powers of p, since p is prime. (Sorry, this part of inner product was vague. The absolute value we have to use here is $|\cdot|_p$ itself.)

(iv) Let $x = p^a n/d$ and $y = p^b m/k$, where n, d, m, k are not divisible by p. Assume for simplicity that $a \le b$. Then,

$$x + y = p^a(n/d + p^{b-a} m/k = p^a \frac{nk + p^{b-a} md}{dk}.$$

Since p is a prime that doesn't divide d or k, it does not divide dk. Let c be the largest integer such that $nk + p^{b-a} md = p^c l$ where l is not divisible by p. Then

$$|x + y|_p = |p^{a+c} \frac{l}{dk}| = p^{-a-c} \le p^{-a} \le \max\{p^{-a}, p^{-b}\} = \max\{|x|_p, |y|_p\} \le |x|_p + |y|_p.$$

(b) Show that $|\cdot|_p$ is non-archimedean with respect to distance, that is,

$$|x + y|_p \le \max\{|x|_p, |y|_p\}.$$

It has been observed that on quantum scales, distances behave this way. (What does this say about triangles in such spaces?)

See above.

(c) Compute $|n/8|_2$ for $1 \le n \le 8$ and verify the non-archimedean property for some case.

$$\begin{bmatrix} |1/8|_2 \\ |2/8|_2 \\ |3/8|_2 \\ |4/8|_2 \\ |5/8|_2 \\ |6/8|_2 \\ |7/8|_2 \\ |8/8|_2 \end{bmatrix} = \begin{bmatrix} |1/8|_2 \\ |1/4|_2 \\ |3/8|_2 \\ |1/2|_2 \\ |5/8|_2 \\ |3/4|_2 \\ |7/8|_2 \\ |1|_2 \end{bmatrix} = \begin{bmatrix} |2^{-3}|_2 \\ |2^{-2}|_2 \\ |2^{-3}3|_2 \\ |2^{-1}|_2 \\ |2^{-3}5|_2 \\ |2^{-2}3|_2 \\ |2^{-3}7|_2 \\ |1|_2 \end{bmatrix} = \begin{bmatrix} 2^3 \\ 2^2 \\ 2^3 \\ 2 \\ 2^3 \\ 2^2 \\ 2^3 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 8 \\ 2 \\ 8 \\ 4 \\ 8 \\ 1 \end{bmatrix}.$$

Notice that

$$4 = |6/8|_p = |4/8 + 2/8|_p \leq \max\{|1/2|_p, |1/4|_p\} = \max\{2, 4\} = 4.$$

Triangles in such spaces have the following property: all triangles are isosceles. Let $|x|_p$ be the length of one side and $|y|_p$ be the length of another side (and therefore the third side has length $|x + y|_p$).

If $|x|_p = |y|_p$ then the triangle is isosceles and we are done. Otherwise, we can assume that $|x|_p > |y|_p$. In this case,

$$|x + y|_p \leq \max\{|x|_p, |y|_p\} = |x|_p.$$

But also by the triangle inequality,

$$|x|_p = |x + y - y|_p \leq \max\{|x + y|_p, |y|_p\}.$$

Since $|x|_p > |y|_p$, the maximum above must be $|x + y|_p$. So $|x|_p \leq |x + y|_p$.

We just showed that $|x + y|_p \leq |x|_p \leq |x + y|_p$ and therefore $|x|_p = |x + y|_p$.