1. (20 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x^2 - y^2 + x^2y + \frac{1}{2}y^2$.

(a) Find and classify the critical points of f.

(b) Let D be the triangle with vertices $(0, 0), (1, 1)$, and $(1, -1)$. Find the absolute maximum and minimum of f on D.

Solution:

(a)
$$f_x = 2x + 2xy \text{ and } f_y = -2y + x^2 + y = x^2 - y.$$

$2x + 2xy = 0$ if $2x(1 + y) = 0$, that is, if **x=0 or y=-1.**

- If $x = 0$, then $f_y(0, y) = -y = 0$ only when $y = 0$. So **(0,0) is a critical point**.
- If $y = -1$, then $f_y(x, -1) = x^2 + 1 = 0$ has no solutions. **Therefore (0,0) is the only critical point**.

Next, taking second derivatives, we get that the Hessian matrix is equal to

$$\begin{bmatrix} 1+y & 2x \\ 2x & -1 \end{bmatrix}_{(0,0)} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The determinant is negative, which means **(0,0) is a saddle point**.

(b) We already found that **(0,0) is the only critical point** (and it's in the region). Next, we need to check the boundaries. The three sides can be parametrized by $y = x$ for $0 \le x \le 1$, $y = -x$ for $0 \le x \le 1$, and $x = 1$ for $-1 \le y \le 1$, respectively.

$y = x$: $f(x, x) = x^3 + \frac{1}{2}x^2 = g_1(x)$. Then $g_1'(x) = 3x^2 + x = 0$ implies $x(3x + 1) = 0$. Since $x = -1/3$ is not in our region, that leaves $x = 0$. Since $y = x$, we also have $y = 0$. But this is already in our list of points to check.

$y = -x$: $f(x, -x) = -x^3 + \frac{1}{2}x^2 = g_2(x)$. Then $g_2'(x) = -3x^2 + x = 0$ implies $x(1 - 3x) = 0$. So $x = 0$ or $x = \frac{1}{3}$. The point $x = 0$ will not give anything new, but $x = 1/3$ gives us the point **(1/3, -1/3)**.

$x = 1$: $f(1, y) = 1 - y^2 + y + \frac{1}{2}y^2 = g_3(y)$. Then $g_3'(y) = -2y + 1 + y = 1 - y$, which equals zero only when $y = 1$. This gives us the point **(1,1)**.

Finally, we have to check the boundary points of the domains of our three functions. These coincide with the vertices; only one vertex gives us a new point: **(1,-1)**. Finally, we check all of the outputs to find that $3/2$ is the maximum and $-1/2$ is the minimum:

$$f(0, 0) = 0, \qquad f(1/3, -1/3) = -\frac{1}{27} + \frac{1}{18} = \frac{1}{54},$$
$$f(1, 1) = 1 + \frac{1}{3} = \frac{3}{2}, \qquad f(1, -1) = -1 + \frac{1}{2} = -\frac{1}{2}.$$

2. (20 points) Define $g : [-\pi, \pi] \to \mathbb{R}$ by $g(x) = \sin(x^3)$. Let C be the graph of g.
Compute $\displaystyle\int_C x^4 y dx + x^5 dy$.
(Hint 1: $x^4 y dx = 5x^4 y dx - 4x^4 y dx$.)
(Hint 2: $\displaystyle\int_{-\pi}^{\pi} x^4 \sin(x^3) dx = 0$.)

Solution: Since the curve C is the image of the graph of the function g, we can parametrize it by

$$\mathbf{r}(t) = \big(t, \sin(t^3)\big), -\pi \leq t \leq \pi.$$

Using the hint, rewrite

$$\int_C x^4 y dx + x^5 dy = \underbrace{\int_C 5x^4 y dx + x^5 dy}_{I_1} - \underbrace{\int_C 4x^4 y dx}_{I_2}.$$

Since the vector field $\mathbf{F}(x, y) = (5x^4 y, x^5)$ is the gradient of $f(x, y) = x^5 y$, we can apply the Fundamental Theorem of Line Integrals. It tells us

$$I_1 = \int_C \mathbf{F} \cdot d\mathbf{s} = f(\mathbf{r}(\pi)) - f(\mathbf{r}(-\pi)) = \pi^5 \sin(\pi^3) - (-\pi)^5 \sin((-\pi)^3) = \pi^5 \sin(\pi^3) - \pi^5 \sin(\pi^3) = 0.$$

For I_2, we parametrize the same way and use the hint:

$$I_2 = 4 \int_C x^4 y dx = 4 \int_{-\pi}^{\pi} x^4 \sin(x^3) dx = 0.$$

Therefore the final answer is 0.

3. (20 points) Let C be the oriented curve shown below[1] where the curved arc is a semicircle and the line is the graph of $y = -x$. Verify Green's Theorem (or Stokes' Theorem) for the integral $\int_C ydx - xdy$.

(Hint: a sign error will only be a 4 point deduction.)

Solution:

(I) First let's use Green's Theorem. Since C is negatively oriented, Green's Theorem says

$$\int_C ydx - xdy = -\iint_D \frac{\partial(-x)}{\partial x} - \frac{\partial y}{\partial y} dA = \iint_D 2dA = \int_{-\pi/4}^{3\pi/4}\int_0^1 2rdrd\theta = \pi \int_0^1 2rdr = \pi.$$

(II) Next, let's break C up into the piecewise curves C_1, the graph of $y = -x$, and C_2, the semicircle. C_1 can be parametrized by $(t, -t)$ for t starting from $\sqrt{2}/2$ and ending at $-\sqrt{2}/2$. So the integral over this region equals

$$\int_{\sqrt{2}/2}^{-\sqrt{2}/2} (-t)dt - t(-dt) = \int_{\sqrt{2}/2}^{-\sqrt{2}/2} 0dt = 0.$$

C_2 can be parametrized by $(\cos(t), \sin(t))$ for t between $3\pi/4$ and $-\pi/4$ (as $(\cos(t), \sin(t)) = (-\sqrt{2}/2, \sqrt{2}/2)$ at $t = 3\pi/4$ and equals $(\sqrt{2}/2, -\sqrt{2}/2)$ at $t = -\pi/4$). The integral then becomes

$$\int_{3\pi/4}^{-\pi/4} \sin(t)(-\sin(t)dt) - \cos(t)(\cos(t)dt) = \int_{-\pi/4}^{3\pi/4} 1dt = \frac{3\pi}{4} - \frac{-\pi}{4} = \pi.$$

So Green's Theorem was verified for this case since

$$\int_C ydx - xdy = \int_{C_1} ydx - xdy + \int_{C_2} ydx - xdy = 0 + \pi = \pi.$$

[1] It was the line from $(\sqrt{2}/2, -\sqrt{2}/2)$ to $(-\sqrt{2}/2, \sqrt{2}/2)$, then a half circle from $(-\sqrt{2}/2, \sqrt{2}/2)$ to $(\sqrt{2}/2, -\sqrt{2}/2)$ that I did not want to spend time compiling with tikz.

4. (20 points)

(a) When are we allowed to apply Divergence Theorem (Gauss's Theorem)?

(b) Let S be the unit sphere $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. Define $\mathbf{F} : S \to \mathbb{R}^3$ by

$$\mathbf{F}(x, y, z) = \left(\frac{x}{(x^2 + y^2 + z^2)^{3/2}}, \frac{y}{(x^2 + y^2 + z^2)^{3/2}}, \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \right).$$

Compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

Solution:

(a) $\mathbf{F}$ needs to be a C^1 vector field defined on the whole region we take the triple integral over. Taking $\mathbf{F}$ as in (b), $\mathbf{F}$ needs to be C^1 on the whole unit ball $x^2 + y^2 + z^2 \leq 1$. Since it's not defined at zero, we are not allowed to use the theorem for part (b) (you can check that the divergence of this vector field $\mathbf{F}$ is simply zero everywhere it's defined).

(b) Recall that we can parametrize the unit sphere by $\mathbf{T}(\theta, \phi) = (\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi)) = (x, y, z)$ for $0 \leq \theta < 2\pi$ and $0 \leq \phi \leq \pi$. Also, we have computed before that in this case the outward pointing normal is

$$\mathbf{T}_\phi \times \mathbf{T}_\theta = (\cos(\theta)\sin^2(\phi), \sin(\theta)\sin^2(\phi), \cos(\phi)\sin(\phi)$$

and therefore

$$d\mathbf{S} = \sin(\phi)(\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi))d\theta d\phi.$$

On the unit sphere, we just have $\mathbf{F} = (x, y, z)$. So $\mathbf{F} \cdot d\mathbf{S} = \sin(\phi)d\theta d\phi$. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^\pi \int_0^{2\pi} \sin(\phi)d\theta d\phi = 2 \cdot 2\pi = 4\pi.$$

Note that this is not zero, so indeed we may not just apply Divergence Theorem without checking the conditions.

5. (20 points) Let E be the closed region in $\mathbb{R}^3$ bounded by $z = x^2 + y^2 - 1$ and $z = 1$. Let $\mathbf{F}$ be the vector field on E defined by $\mathbf{F}(x, y, z) = (x, y, z + 1)$.

Verify the Divergence Theorem for the integral $\displaystyle\iint_{\partial E} \mathbf{F} \cdot d\mathbf{S}$.

Solution:

(I) Using the theorem, we get

$$\iint_{\partial E} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F} dV = \iiint_E 3dV.$$

This region is probably not simple enough for us to have the volume memorized. The top function is $z = 1$ and the bottom function is $z = x^2 + y^2 - 1$. Setting them equal, we see that the boundary of the $dxdy$ region is $x^2 + y^2 = 2$. So the integral becomes

$$\iiint dV = \iint_{x^2+y^2\leq 2} 1 - (x^2 + y^2 - 1)dA = \int_0^{2\pi} \int_0^{\sqrt{2}} (2 - r^2)rdrd\theta = 2\pi \int_0^{\sqrt{2}} 2r - r^3 dr$$
$$= 2\pi \left(r^2 - \frac{r^4}{4} \right) \Big|_0^{\sqrt{2}} = 2\pi(2 - 1) = 2\pi.$$

Multiplying by the scalar 3 originally inside the integral, the answer is 6π.

(II) Computing directly, we should probably parametrize both C^1 surfaces by functions. The region of integration for both will still be $x^2 + y^2 \leq 2$. Since $z = 1$ is the top function, we can parametrize by $(x, y, 1)$ to get $(0, 0, 1)$ as the normal vector. This is the correct vector since it is the outward pointing one in this case. The vector field $\mathbf{F}$ becomes $(x, y, 2)$ on this surface. Plugging everything in, we get

$$\iint_{z=1} \mathbf{F} \cdot d\mathbf{S} = \iint_{x^2+y^2\leq 2} (x, y, 2) \cdot (0, 0, 1)dxdy = \iint_{x^2+y^2\leq 2} 2dA = 2(\text{Area of circle with radius } \sqrt{2}),$$

which is $2 \cdot \pi 2 = 4\pi$.

The second surface can be parametrized by $(x, y, x^2 + y^2 - 1)$, so the normal vector is $(-2x, -2y, 1)$. However, this is the inward pointing normal, so we need to multiply by -1 to get $(2x, 2y, -1)$. On this surface, $\mathbf{F} = (x, y, x^2 + y^2)$. So

$$\iint_{z=x^2+y^2-1} \mathbf{F} \cdot d\mathbf{S} = \iint_{x^2+y^2\leq 2} (x, y, x^2 + y^2) \cdot (2x, 2y, -1)dxdy = \iint_{x^2+y^2\leq 2} x^2 + y^2 dxdy.$$

Converting to polar, the integral becomes

$$\int_0^{2\pi} \int_0^{\sqrt{2}} r^2 rdrd\theta = 2\pi \int_0^{\sqrt{2}} r^3 dr = 2\pi \frac{(\sqrt{2})^4}{4} = 2\pi.$$

Finally, $4\pi + 2\pi = 6\pi$.

6. (30 points) Define

$$C_3 = \{(u, v, w) \in \mathbb{R}^3 : 0 < u, v, w < 1\},$$

and

$$T_3 = \{(x, y, z) \in \mathbb{R}^3 : 0 < x + y + z < 1, \text{ and } x, y, z > 0\}.$$

That is, C_3 is an open unit cube in $\mathbb{R}^3$ and T_3 is an open tetrahedron in $\mathbb{R}^3$.

Define $\Phi : C_3 \to T_3$ by $\Phi(u, v, w) = (x, y, z)$, where

$$\begin{aligned} x &= u, \\ y &= (1-u)v, \\ z &= (1-u)(1-v)w. \end{aligned}$$

(a) Show that Φ is injective. Recall: Φ is injective if $\Phi(u, v, w) = \Phi(u', v', w')$ implies $(u, v, w) = (u', v', w')$.

(b) Show that Φ is surjective. Recall: Φ is surjective if for all points $(x, y, z) \in T_3$ there is some $(u, v, w) \in C_3$ such that $\Phi(u, v, w) = (x, y, z)$.

(Hint: you might first want to show that $x + y + z = 1 - (1-u)(1-v)(1-w)$, and then use the definition of C_3 and I_3.)

(c) Find the volume of T_3 by applying the change of variables Φ to the volume integral $\iiint_{T_3} dV$.

Solution:

(a) Assume $\Phi(u, v, w) = \Phi(u', v', w')$. Then by definition of Φ,

$$\begin{aligned} u &= u', \\ (1-u)v &= (1-u')v', \\ (1-u)(1-v)w &= (1-u')(1-v')w'. \end{aligned}$$

From the first equation, $u = u'$. So the second equation becomes $(1-u)v = (1-u)v'$. Since $u \neq 1$ we can divide and get $v = v'$. Repeating the same procedure, $w = w'$. Therefore $(u, v, w) = (u', v', w')$ and we are done checking the injectivity of Φ.

(b) We use the hint. First,

$$\begin{aligned} x + y + z = u + (1-u)v + (1-u)(1-v)w &= u + v - uv + (1 - u - v + uv)w \\ &= (u + v + w) - (uv + uw + vw) + uvw. \end{aligned}$$

Next,

$$\begin{aligned} 1 - (1-u)(1-v)(1-w) = 1 - (1 - u - v + uv)(1-w) &= 1 - (1 - u - v + uv - w + uw + vw - uvw) \\ &= (u + v + w) - (uv + uw + vw) + uvw. \end{aligned}$$

Therefore $x + y + z = 1 - (1-u)(1-v)(1-w)$. Since $0 < a < 1$ implies $0 < 1 - a < 1$, we conclude $u, v, w, (1-u), (1-v), (1-w)$ are all strictly between 0 and 1, and so $0 < 1-(1-u)(1-v)(1-w) < 1$, proving that $0 < x + y + z < 1$. Moreover, $x, y, z > 0$ because of their definition and the same reason.

Finally, we can solve for $u = x$; $y = (1-u)v$, so $v = y/(1-x)$; and $z = (1-u)(1-v)w$, so $w = z/[(1-x)(1-y)]$. Therefore Φ has a well-defined inverse taking the set $x, y, z > 0$ with $0 < x+y+z < 1$ to the unit cube, so it is onto.

(c) The point of (a) and (b) is to understand how the change of variables works, or else we would not know our new region under this change, and might not know it's one-to-one (injective) and therefore this change of variables would not locally preserve area. It is *not* enough to check the invertibility of the Jacobian because Φ is definitely not a linear change of variables (it takes a cube to a tetrahedron, not a parallelogram). Now that we know the image of Φ and that we are allowed to use this change of variables, we can compute the volume of the tetrahedron:

$$\iiint_{T_3} dV = \iiint_{T_3} dxdydz = \iiint_{C_3} \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw. \tag{1}$$

By the definition of x, y, z above, the Jacobian is just

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -v & (1-u) & 0 \\ -(1-v)w & -(1-u)w & (1-u)(1-v) \end{bmatrix}.$$

Since this matrix is triangular, the determinant is the product of the diagonal entries, namely, $|\frac{\partial(x,y,z)}{\partial(u,v,w)}| = (1-u)^2(1-v)$. (We don't need to take absolute values in this case since this determinant is positive.) Then (1) becomes

$$\int_0^1 \int_0^1 \int_0^1 (1-u)^2(1-v) dudvdw = \frac{-(1-u)^3}{3}\Big|_0^1 \cdot \frac{-(1-v)^2}{2}\Big|_0^1 \cdot w\Big|_0^1 = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}.$$

You can apply the same method for "tetrahedrons" in $\mathbb{R}^n$ and get the volume is $1/n!$. In $\mathbb{R}^3$ the above is more work than setting up the integral directly, but in higher dimensions this change of variables is much simpler.[2]

[2] See *Principles of Mathematical Analysis*, Walter Rudin, chapter 10, problem 13.