DESCRIPTION OF RESEARCH

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My primary research interests are in algebraic topology and algebraic K-theory. In particular, I use the tools of equivariant stable homotopy theory to study algebraic K-theory and related invariants. Algebraic K-theory associates to a ring A a sequence of groups $K_i(A)$. In general these groups are very difficult to compute. Despite the fact that higher algebraic K-theory was defined over 40 years ago, even for basic rings like $\mathbb{Z}/p^2$, the K-theory groups aren't completely known. Interest in K-theory computations remains strong, however, as K-theory is an invariant which touches on many mathematical fields. Algebraic K-theory lies in the intersection of algebraic topology, algebraic geometry, and number theory, with applications to motivic homotopy theory, classification of manifolds, special values of L-functions, etc. One fruitful approach to the study of algebraic K-theory uses the tools of equivariant stable homotopy theory. Equivariant stable homotopy theory is a branch of algebraic topology which studies objects (spectra) with an action of a compact Lie group [46], [47]. Although algebraic K-theory is not an equivariant object, it is the tools of equivariant stable homotopy theory that can often most easily facilitate its computation. This is a remarkable fact about equivariant stable homotopy theory: that it can be used with great success to study questions which on the surface are not equvariant.

Below I will outline some of my recent results as well as several current projects. My research falls into two main categories:

- Computing algebraic K-theory groups which were previously inaccessible, using recent results and new methods from equivariant stable homotopy theory.
- Developing the theory around related tools such as topological Hochschild homology, topological coHochschild homology, and equivariant algebraic structures that arise in K-theory computations.

Before describing my individual research projects, I would like to recall the equivariant stable homotopy approach to algebraic K-theory. Algebraic K-groups are difficult to compute but they can be approached via invariants which are more computable. Let A be a ring. There is map between the algebraic K-theory of A and ordinary Hochschild homology of A, called the Dennis trace map:

$$K_i(A) \to HH_i(A).$$

In the 1980's Goodwillie [32] proved that the Dennis trace map lifts through negative cyclic homology,

$$K_i(A) \to HC_q^-(A) \to HH_i(A),$$

and that rationally $HC_q^-(A)$ is a good approximation to algebraic K-theory. This led to computations of the ranks of various algebraic K-groups (for example, in [30]). In his ICM address in 1990, Goodwillie conjectured that there should be topological analogs of Hochschild homology and cyclic homology, defined by replacing the ground ring $\mathbb{Z}$ with the sphere spectrum. The hope was that such topological analogs would yield information about the torsion in algebraic K-theory. The topological version of Hochschild homology (THH) was defined by Bökstedt [15] (and implicitly Breen [16]). At that time, modern symmetric monoidal categories of spectra had not yet been invented, so Bökstedt developed coherence machinery to make sense of the conceptual picture that one should replace tensor product over $\mathbb{Z}$ with smash products over the sphere spectrum to get from algebraic Hochschild homology to topological Hochschild homology. We return to discuss Bökstedt's construction of THH, and subsequent developments in Section 2.1. Bökstedt also defined a topological version of the Dennis trace map:

$$tr : K_q(A) \to \mathrm{THH}_q(A).$$

To complete this story, Bökstedt-Hsiang-Madsen [14] defined topological cyclic homology (TC), a topological refinement of Connes' cyclic homology. Further, they constructed the cyclotomic trace map from algebraic K-theory to topological cyclic homology, lifting the topological Dennis trace:

$$K(A) \to TC(A) \to THH(A).$$

When working at a prime p, topological cyclic homology can be a very good approximation to algebraic K-theory. Indeed, a number of authors have proven such comparison results (see, for example, [48], [23], [29]). Thus, to understand $K(A)$, one would first like to understand $\mathrm{TC}(A)$.

The definition of topological cyclic homology relies on an equivariant structure on $\mathrm{THH}(A)$. Topological Hochschild homology is the geometric realization of a cyclic spectrum, and thus has a natural S^1-action by the theory of cyclic sets ([20], [18], [43]). Topological Hochschild homology can be constructed as a genuine S^1-spectrum with what is called a cyclotomic structure [40]. Topological cyclic homology is then defined from the fixed points of topological Hochschild homology. Because these steps are essential to the computation of topological cyclic homology, and hence K-theory, I would like to make this more precise. The homotopy groups of the fixed point spectra of THH are called TR-groups. More explicitly, for a ring A and a fixed prime p,

$$\mathrm{TR}^n_q(A;p) = \pi_q(\mathrm{THH}(A)^{C_{p^{n-1}}}) = [S^q \wedge S^1/C_{p^{n-1}}{}_+, \mathrm{THH}(A)]_{S^1}$$

where $\mathrm{THH}(A)$ denotes the topological Hochschild S^1-spectrum of A and $\mathrm{THH}(A)^{C_{p^{n-1}}}$ denotes the $C_{p^{n-1}}$ fixed points of this spectrum. These TR-groups come equipped with several operators and relations which provide a rigid algebraic structure, making computations possible. We will discuss this structure more in Sections 2.3 and 2.6. Topological cyclic homology is defined by a homotopy limit construction that involves these operators. Thus understanding the TR-groups of a ring and the associated operators makes it possible to compute the topological cyclic homology of the ring, and hence its algebraic K-theory. The above discussion applies to studying the K-theory of any ring spectrum R, not just the Eilenberg-Maclane spectra of rings, but we will focus our discussion on the case of rings. Note, it is only possible to define topological cyclic homology if one starts with a model of topological Hochschild homology as a cyclotomic S^1-equivariant spectrum. This point is relevant to the later discussion in Section 2.1.

One vein of my research is that I combine the above approach with modern methods and new computations in equivariant stable homotopy theory to compute algebraic K-groups which were previously inaccessible. Early in my career I worked primarily on such computation questions, and have had a variety of successes as outlined in Sections 1.1, 1.2, 1.3, and 1.4 below. A recent and ongoing project in this vein in described in Section 1.5. In recent years I have expanded my research to include a variety of projects developing foundations and new results for the related invariants: topological Hochschild homology, topological coHochschild homology, topological cyclic homology, etc. Sections 2.1, 2.2, 2.3, 2.4, 2.5, and 2.6 describe my work in these areas. Throughout the discussion, ongoing work and future directions are identified.

1. Algebraic K-Theory Computations

In this section I will describe several past and current projects focused on the computation of algebraic K-theory groups. In particular I will describe several projects computing the algebraic K-groups of pointed monoid algebras (for example $K(\mathbb{Z}[x]/x^2), K(\mathbb{Z}[x,y]/(xy)), K(\mathbb{Z}[C_2])$, etc).

Given a ring A and a pointed monoid Π, let $A(\Pi)$ denote the corresponding pointed monoid algebra. As explained earlier, to apply the homotopy-theoretic method described above, we first need to understand the topological Hochschild homology $\mathrm{THH}(A(\Pi))$. There is an equivalence of S^1-spectra [39]

$$\mathrm{THH}(A) \wedge N^{cy}(\Pi) \xrightarrow{\simeq} \mathrm{THH}(A(\Pi)),$$

where $N^{cy}(\Pi)$ denotes the cyclic bar construction on Π. In order to compute the topological cyclic homology of pointed monoid algebras, we first need to look at the fixed points of topological Hochschild homology, or TR-groups. Recall that

$$\mathrm{TR}^n_q(A(\Pi);p) = \pi_q(\mathrm{THH}(A(\Pi))^{C_{p^{n-1}}}) = [S^q \wedge S^1/C_{p^{n-1}}{}_+, \mathrm{THH}(A(\Pi))]_{S^1}.$$

So by the equivalence above,

$$\mathrm{TR}^n_q(A(\Pi);p) \cong [S^q \wedge S^1/C_{p^{n-1}}{}_+, \mathrm{THH}(A) \wedge N^{cy}(\Pi)]_{S^1}.$$

In order to compute these groups, we must understand not only the ordinary homotopy type of the cyclic bar construction $N^{cy}(\Pi)$, but the S^1-equivariant homotopy type. In other words, we must specify how to build this equivariant homotopy type out of representation spheres S^λ. Here λ is a finite-dimensional real representation of S^1, and S^λ denotes the one point compactification of this representation. If we replace the cyclic bar construction by these representation spheres in the TR-groups above, we get many groups of the form:

$$\mathrm{TR}^n_{q-\lambda}(A;p) \cong [S^q \wedge S^1/C_{p^{n-1}}{}_+, \mathrm{THH}(A) \wedge S^\lambda]_{S^1}.$$

There are called $RO(S^1)$-graded TR-groups of A. In other words, they are the $RO(S^1)$-graded equivariant homotopy groups of $\mathrm{THH}(A)$. Note that we have made a trade-off. We were aiming to compute ordinary TR-groups of a pointed monoid algebra $A(\Pi)$, and instead we replaced them with $RO(S^1)$-graded TR-groups of the simpler ring A. This is only a good trade-off if it is possible to compute these equivariant homotopy groups. Section 1.1 below describes work I have done computing such $RO(S^1)$-graded TR-groups. Sections 1.2, 1.3, 1.4, and 1.5 describe new algebraic K-theory computations that were possible because of these $RO(S^1)$-graded TR computations.

1.1. $RO(S^1)$-graded TR of $\mathbb{F}_p$, $\mathbb{Z}$, and ℓ.

The topological Hochschild S^1-spectrum has naturally associated equivariant homotopy groups which give a TR-theory graded by the real representation ring of the circle, $RO(S^1)$. These groups arise naturally when computing algebraic K-theory, as described above. Elements in the representation ring are given by formal differences of isomorphism classes of representations. For every $\alpha \in RO(S^1)$, we choose representatives β and γ such that $\alpha = [\beta] - [\gamma]$. Let S^β denote the one-point compactification of the representation β. Then the $RO(S^1)$-graded TR-groups are defined as the $RO(S^1)$-graded equivariant homotopy groups

$$\mathrm{TR}^n_\alpha(A;p) = \pi_\alpha(THH(A)^{C_{p^{n-1}}}) = [S^\beta \wedge S^1/C_{p^{n-1}}{}_+, \mathrm{THH}(A) \wedge S^\gamma]_{S^1}.$$

These $RO(S^1)$-graded TR-groups first arose in computations of the algebraic K-theory of singular varieties. One motivation for studying the K-theory of singular varieties comes from algebraic geometry. Algebraic K-theory is closely related to motivic homotopy theory, a field that uses homotopy theoretic methods to solve problems in algebraic geometry. In particular, algebraic K-theory is the analog in the motivic world of topological K-theory in the topology setting. For example, there is an Atiyah Hirzebruch-type spectral sequence from motivic cohomology to algebraic K-theory for any regular scheme ([10], [11], [28], [50]). However, motivic cohomology for singular varieties is less well understood. Because of its close relationship with motivic cohomology, understanding the K-theory of singular varieties is an important step towards fully understanding motivic cohomology in these cases.

One of the first fully computed examples of the algebraic K-theory of a singular variety is due to Hesselholt and Madsen [39]. For an $\mathbb{F}_p$-algebra A, Hesselholt and Madsen computed the K-theory of $A[x]/(x^m)$. They did this by expressing $K(A[x]/(x^m))$ in terms of the $RO(S^1)$-graded TR-groups of A. In [40] they computed these $RO(S^1)$-graded TR-groups in the case where the virtual representation α was of the form $\alpha = 2q - \lambda$ with $q \in \mathbb{Z}$ and $\lambda \in R(S^1)$ a complex S^1-representation. These were the particular groups necessary for that algebraic K-theory computation.

This naturally leads to two questions. One, is it possible to fully compute the $RO(S^1)$-graded TR-groups $TR^n_\alpha(\mathbb{F}_p; p)$ for all virtual representations α? And two, is there any hope of computing the $RO(S^1)$-graded TR-groups of rings such as the integers? The latter would allow for the computation of the algebraic K-theory of some pointed monoid algebras $\mathbb{Z}(\Pi)$. It turns out that the answer to both questions is yes. In [31], I computed $\mathrm{TR}^n_\alpha(\mathbb{F}_p; p)$ for all n and all even-dimensional representations α. I continued this work in a joint project with Vigleik Angeltveit [6] where we completed the picture for $\mathbb{F}_p$ by also computing the $RO(S^1)$-graded TR-groups of $\mathbb{F}_p$ for even dimensional virtual representations α. We then continued on to study the $RO(S^1)$-graded TR-groups of $\mathbb{Z}$ and the Adams summand ℓ of connective complex K-theory.

To do these computations we developed a spectral sequence which converges to $\mathrm{TR}^n_{\alpha+*}(R; p, V)$ for a ring R, a finite complex V, and a virtual representation $\alpha \in RO(S^1)$. We computed the differentials in this spectral sequence, allowing us to explicitly compute all of the $RO(S^1)$-graded TR-groups of $\mathbb{F}_p$, including the previously undetermined groups graded by odd dimensional representations. Further, these techniques allowed us to compute $\mathrm{TR}^n_\alpha(\mathbb{Z}; p, \mathbb{Z}/p)$ and $\mathrm{TR}^n_\alpha(\ell; p, V(1))$, the $RO(S^1)$-graded TR-groups of $\mathbb{Z}$ with $\mathbb{Z}/p$-coefficients, and the $RO(S^1)$-graded TR-groups of the Adams summand ℓ with $V(1)$-coefficients. These computations for $\mathbb{Z}$ were essential to the algebraic K-theory results described Sections 1.2, 1.3, 1.4, and 1.5 below.

An interesting question coming out of this work is whether we can understand the abstract algebraic structure that these $RO(S^1)$-graded TR-groups fit into. Ordinary TR-groups for a ring A are an example of a rigid algebraic structure called a Witt complex. This structure, and the relationship with the initial object in the category of Witt complex over A, the de Rham-Witt complex of A, facilitates ordinary TR-computations. Therefore it would be very interesting to understand the $RO(S^1)$-graded analogs of these structures. This is discussed in Section 2.6.

1.2. The algebraic K-theory of truncated polynomial algebras over $\mathbb{Z}$.

About 35 years ago, Soulé [49] studied the algebraic K theory of the dual numbers, $K_q(\mathbb{Z}[x]/(x^2))$. This K-theory group splits as

$$K_q(\mathbb{Z}[x]/(x^2)) \cong K_q(\mathbb{Z}) \oplus K_q(\mathbb{Z}[x]/(x^2), (x)),$$

where the latter group denotes relative algebraic K-theory. As mentioned above, in the early days of higher algebraic K-theory computations, often the rank of the groups was accessible, while the torsion was not. Indeed, in this case Soulé proved that the relative algebraic K-theory group $K_q(\mathbb{Z}[x]/(x^2),(x))$ is a finitely generated abelian group of rank 1 if q is odd and 0 if q is even. With Soulé's techniques, however, he was unable to determine whether there is torsion in these groups. In joint work with Vigleik Angeltveit and Lars Hesseltholt [8] we proved the following

Theorem. *Let m be a positive integer and i a non-negative integer. Then*

1. *The abelian group $K_{2i+1}(\mathbb{Z}[x]/x^m),(x))$ is free of rank $m-1$*

2. *The abelian group $K_{2i}(\mathbb{Z}[x]/x^m),(x))$ is finite of order $(mi)!(i!)^{m-2}$.*

To prove this, we used the computational method outlined above for the K-theory of pointed monoid algebras. This method allowed us to rewrite these K-theory groups in terms of the $RO(S^1)$-graded TR-groups of $\mathbb{Z}$. Then, using our computations of $\mathrm{TR}^n_\alpha(\mathbb{Z};p,\mathbb{Z}/p)$ mentioned in Section 1.1 above, we were able to determine enough information about $\mathrm{TR}^n_\alpha(\mathbb{Z};p)$ to prove the theorem.

1.3. The algebraic K-theory of the coordinate axes over the integers.

In [37] Lars Hesselholt computed the relative algebraic K-theory of the coordinate axes over $\mathbb{F}_p$, $K_q(\mathbb{F}_p[x,y]/(xy),(x,y))$. The analogous computation over the integers has been considered by various authors over the years. In the 1970's Dennis and Krusemeyer [21] computed $K_2(\mathbb{Z}[x,y]/(xy),(x,y)) \cong \mathbb{Z}$ using the classical algebraic definition of K_2. In the 1980's Geller, Reid, and Weibel [30] showed that for every nonnegative integer q the abelian group $K_q(\mathbb{Z}[x,y]/(xy),(x,y))$ has rank 0 if q is odd and 1 if q is even. Nothing was known, however, about the torsion of the groups for $q > 2$. In joint work with Vigleik Angeltveit [5], we proved:

Theorem. *For any $i \geq 0$,*

1. *The abelian group $K_{2i}(\mathbb{Z}[x,y]/(xy),(x,y))$ is free of rank 1.*

2. *The abelian group $K_{2i+1}(\mathbb{Z}[x,y]/(xy),(x,y))$ is finite of order $(i!)^2$.*

Our work improves on the past results by using the full power of modern methods in equivariant homotopy theory. In particular, we used the trace method for computing the algebraic K-theory of a pointed monoid algebra, along wtih the new $RO(S^1)$-graded TR computations for $\mathbb{Z}$ described in Section 1.1.

1.4. The algebraic K-theory of truncated polynomials in several variables.

In [39] Lars Hesselholt and Ib Madsen computed the algebraic K-theory groups $K_q(k[x]/x^m,(x))$, for k a perfect field of characteristic p>0. In Section 1.2 above, we discussed the analog of this result over the integers. Another natural extension of this work is to compute the algebraic K-theory groups of truncated polynomial algebras in several commuting variables. In joint work with Vigleik Angletveit, Michael Hill, and Ayelet Lindenstrauss [9], for k a perfect field of characteristic p>0, we computed the groups

$$K_q(k[x_1,x_2,\dots x_n]/(x_1^{a_1},x_2^{a_2},\dots x_n^{a_n}))$$

in the case where $p \nmid a_i$ for all $1 \leq i \leq n$. We also prove rational results for $K_q(\mathbb{Z}[x_1,\dots x_n]/(x_1^{a_1},\dots x_n^{a_n}))$. We prove these results through a careful study of n-cubes of cyclotomic spectra. Whereas in the single variable case (i.e. $n=1$), the result was expressed by Hesselholt and Madsen in terms of the big Witt vectors on k, in the multivariable case a new theory of Witt vectors is needed to organize the result of the computation. In our work we define a new generalization of Witt vectors, called Witt vectors on $\mathbb{N}^n$, and express the K-theory result in terms of these new Witt vectors.

This generalization of Witt vectors is likely to be useful in the computation of other algebraic K-theory groups, just as the ordinary big Witt vectors appear in many K-theory computations. Further, other authors have already begun exploring the relationship between our new theory of Witt vectors and, for instance, formal group laws. In particular, while traditional Witt vectors represent the curves functor taking a formal group G to maps of formal schemes $\mathbb{A}^1 \to G$ ([19], [34]), in [52] Kirsten Wickelgren has shown that analogously the additive formal group underlying our new Witt vectors on a particular truncation set represents the functor taking a formal group G to the maps of formal schemes $\hat{\mathbb{A}}^n \to G$.

1.5. **Algebraic K-theory of $\mathbb{Z}[C_2]$.**

There is great interest in the algebraic K-theory of group rings. Indeed these K-theory groups are the target of the assembly maps in the famous Farrell-Jones conjecture [26], [27]. However, these K-theory groups are very difficult to compute. For instance, the groups $K_i(\mathbb{Z}[C_2])$ are only known for $i = 0, 1$ and 2, and those computations were all done with algebraic definitions of lower algebraic K-theory (e.g. [24]), and thus don't extend to higher algebraic K-theory.

Consider the commutative square

$$\begin{array}{ccc} K(A) & \xrightarrow{t \mapsto 1} & K(\mathbb{Z}) \\ {\scriptstyle t\mapsto -1}\downarrow & & \downarrow \\ K(\mathbb{Z}) & \longrightarrow & K(\mathbb{Z}/2) \end{array}$$

The birelative K-theory spectrum $K(\mathbb{Z}[C_2], \mathbb{Z}, \mathbb{Z})$ is defined as the iterated homotopy fiber of this square. This birelative K-theory measures the failure of $K(\mathbb{Z}[C_2])$ to satisfy excision for rings. In particular, there is a long exact sequence

$$\ldots \to K_i(\mathbb{Z}[C_2], \mathbb{Z}, \mathbb{Z}) \to K_i(\mathbb{Z}[C_2]) \to KH_i(\mathbb{Z}[C_2]) \to K_{i-1}(\mathbb{Z}[C_2], \mathbb{Z}, \mathbb{Z}) \to \ldots$$

where $KH_i(\mathbb{Z}[C_2])$ denotes Weibel's homotopy K-theory [51]. Homotopy K-theory does satisfy excision for rings, and therefore $KH_i(\mathbb{Z}[C_2])$ is much more computable than $K_i(\mathbb{Z}[C_2])$. So the missing piece of the computation is the birelative theory $K(\mathbb{Z}[C_2], \mathbb{Z}, \mathbb{Z})$. The cyclotomic trace map here $K_i(\mathbb{Z}[C_2], \mathbb{Z}, \mathbb{Z}; 2) \to \mathrm{TC}_i(\mathbb{Z}[C_2], \mathbb{Z}, \mathbb{Z})$ is an isomorphism [29]. So we aim to compute the birelative topological cyclic homology $\mathrm{TC}_i(\mathbb{Z}[C_2], \mathbb{Z}, \mathbb{Z})$.

The first step to computing this is to understand the topological Hochschild homology of $\mathbb{Z}[C_2]$. Note that $\mathbb{Z}[C_2]$ is a pointed monoid algebra for the monoid $C_{2+} = \{0, 1, x\}$, where $x^2 = 1$. As explained earlier in the outline for computing the algebraic K-theory of pointed monoid algebras, there is therefore a splitting:

$$\mathrm{THH}(\mathbb{Z}[C_2]) \simeq \mathrm{THH}(\mathbb{Z}) \wedge N^{cy}(C_{2+}).$$

In order to proceed one needs to understand the S^1- equivariant homotopy type of the cyclic bar construction, which is in general very difficult. Indeed there are various algebraic K-theory computations where all of the pieces are in place except for an equivariant understanding of $N^{cy}(\Pi)$ for some monoid Π (see, for example, [38]).

In our case, direct attempts by authors in the past to compute the equivariant homotopy type of $N^{cy}(C_{2+})$ have been unsuccessful. In joint work with Vigleik Angeltveit, we developed a new strategy to attack this problem. We compare the cyclic bar construction $N^{cy}(C_{2+})$ with $N^{cy}(\Pi_2)$, where Π_2 is the pointed monoid $\Pi_2 = \{0, 1, x\}$ with $x^2 = 0$. The equivariant homotopy type of the latter cyclic bar construction is understood by previous work of Hesselholt and Madsen [39]. To compare the two cyclic bar constructions, we define an increasing filtration

$$F^0 N^{cy}(C_{2+}) \to F^1 N^{cy}(C_{2+}) \to F^2 N^{cy}(C_{2+}) \to \dots$$

where $F^s N^{cy}(C_{2+})$ is the geometric realization of the simplicial subcomplex of $N^{cy}(C_{2+})$ of x-degree $\leq s$. Note that the filtration quotients:

$$Gr^s N^{cy}(C_{2+}) = F^s N^{cy}(C_{2+})/F^{s-1} N^{cy}(C_{2+})$$

are homogeneous in x and $x^2 = 0$. In particular $Gr^s N^{cy}(C_{2+}) = N^{cy}(\Pi_2; s)$, where $N^{cy}(\Pi_2; s)$ denotes the x-degree s piece of $N^{cy}(\Pi_2)$, a cyclic bar construction whose equivariant homotopy type we understand. This filtration yields spectral sequences computing the topological Hochschild homology and TR-groups of $\mathbb{Z}[C_2]$, which have E_1-terms which are understood by work of myself with Angeltveit and Hesselholt [8].

In [7] we exploit this new strategy to make low-dimensional calculations of the algebraic K-groups of $\mathbb{Z}[C_2]$. We can recover the groups $K_i(\mathbb{Z}[C_2]$ for $i = 0, 1, 2$, known by classical algebraic methods, as well as getting new results about other lower K-groups. For instance, we prove:

Theorem. *We make the following computations of algebraic K-theory groups:*

$K_0(\mathbb{Z}[C_2]) = \mathbb{Z}$
$K_1(\mathbb{Z}[C_2]) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$
$K_2(\mathbb{Z}[C_2]) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$
$K_3(\mathbb{Z}[C_2])$ *is an extension of* $\mathbb{Z}/8 \oplus \mathbb{Z}/16 \oplus \mathbb{Z}/3$ *by* $\mathbb{Z}/2$*.*
$K_5(\mathbb{Z}[C_2]) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus H$ *where* $|H||16$

In ongoing work, we are in the process of using this new approach to make further computations of low dimensional K-groups of $\mathbb{Z}[C_2]$. We are also working towards more general results, trying to identify K-theory classes in odd algebraic K-groups $K_{2q+1}(\mathbb{Z}[C_2])$ using this approach.

2. Topological Hochschild Homology and Related Tools

In addition to the algebraic K-theory projects described above, in recent years I have also branched out to work on projects developing the theory and applications of related tools such as topological Hochschild homology, topological coHochschild homology, equivariant structures, etc. In the sections below I will describe several of my recent and ongoing projects in these areas.

2.1. New construction of topological Hochschild homology as an equivariant spectrum.

Bökstedt [15] introduced topological Hochschild homology before the invention of symmetric monoidal categories of spectra. Consequently, he had to invent coherence machinery to handle the smash product and then prove that his model of smash powers had the right homotopy type. Later, after the development of modern categories of spectra, it became possible to define topological Hochschild homology with a bar construction, avoiding the complexity of the Bökstedt smash product (for example, in [25]). However, it was long believed that the new approaches in modern categories of spectra did not yield the correct fixed points. Hence anytime one wanted to consider topological Hochschild homology as an equivariant spectrum, one needed to rely on the Bökstedt model. As discussed earlier, to use THH to study algebraic K-theory, it is essential to consider THH as an S^1-equivariant spectrum.

In a joint project of myself, Vigleik Angeltveit, Andrew Blumberg, Michael Hill, Tyler Lawson, and Michael Mandell, we set out to use recent developments in equivariant stable homotopy theory to develop a new equivariant model for topological Hochschild homology. In Hill-Hopkins-Ravenel's proof of the Kervaire Invariant One problem [42], they developed the theory of a multiplicative norm functor N_H^G from H-spectra to G-spectra (which also appeared in work of Greenlees and May [33]). The behavior of this norm functor with respect to geometric fixed points suggests that it is related to the Bökstedt smash product. Indeed, in [2] we construct an explicit comparison between the two as equivariant spectra:

Theorem. *Let X be a cofibrant orthogonal spectrum, and let $\tilde{X}$ denote the underlying symmetric spectrum. Then there is an isomoprhism in the homotopy category of C_k-equivariant orthogonal spectra:*

$$\tilde{X}^{\overset{B}{\wedge} k} \cong N_e^{C_k} X,$$

between the Bökstedt smash product and the Hill-Hopkins-Ravenel norm.

Let R denote a ring spectrum. In a modern symmetric monoidal category of spectra one can define topological Hochschild homology using a bar construction. However, as mentioned above, the equivariant spectra given by such definitions did not seem to be cyclotomic, and thus topological cyclic homology could not be defined from these models of THH. In [3] we introduced a new approach to the construction of the cyclotomic structure on THH, using an interpretation of THH in terms of the Hill-Hopkins-Ravenel norm. In particular we extend the norm construction from finite groups to S^1, defining a functor $N_e^{S^1}$ which sends a ring spectrum R to $\mathcal{I}_{\mathbb{R}^\infty}^{U}|N_\wedge^{cy} R|$. Here $\mathcal{I}_{\mathbb{R}^\infty}^{U}$ denotes a change of universe functor. We then prove that this functor does indeed give a construction of THH as a cyclotomic spectrum.

Theorem. *Let R be a cofibrant associative or cofibrant commutative ring orthogonal spectrum. Then $N_e^{S^1} R$ has a natural structure of a cyclotomic spectrum.*

Another advantage of this model is that it allows us to construct C_n-relative versions of topological Hochschild homology, written $N_{C_n}^{S^1} R$. Further, for R an A-algebra, where A is a cofibrant commutative ring spectrum, we consider an A-relative topological Hochschild homology, ${}_A N_e^{S^1} R$. We prove some conditions under which this also has a cyclotomic type structure.

We are using this new model in ongoing work on the topological Hochschild homology of Thom spectra as discussed in Section 2.2 below. In other ongoing work we aim to understand the algebraic structure of the C_n-relative THH, as discussed in Section 2.3 below.

2.2. **Topological Hochschild homology of Thom spectra.**

In [12], Blumberg-Cohen-Schlichtkrull describe the topological Hochschild homology of ring spectra which are Thom spectra for maps $f : X \to BF$, loop maps to the classifying space of stable spherical fibrations. They identify this topological Hochschild homology (non-equivariantly) as the Thom spectrum of a stable bundle over the free loop space $L(BX)$. In an ongoing joint project with Vigleik Angeltveit, Andrew Blumberg, Michael Hill, Tyler Lawson, and Michael Mandell, we prove an analogous result which also captures the equivariant (cyclotomic) structure of THH. In particular, we use the new construction described in Section 2.1 of topological Hochschild homology as an equivariant spectrum to identify the equivariant homotopy type of THH of a Thom spectrum as an equivariant Thom spectrum [4].

2.3. **An algebraic model of TR.**

As discussed earlier, an essential component to the trace method approach to studying algebraic K-theory is to consider the $C_k \subset S^1$ fixed points of topological Hochschild homology. In a joint project with Vigleik Angeltveit, Andrew Blumberg, Michael Hill, and Tyler Lawson, we look at an algebraic analog of this situation [1]. In particular, we develop a G-equivariant Hochschild homology of an associative Green functor for G. This equivariant version of Hochschild homology is defined as the homology of a twisted Hochschild complex. We also describe geometric fixed points in the derived category of Mackey functors, allowing us to prove that the twisted Hochschild complex has a type of cyclotomic structure.

By work of Hesselholt-Madsen [40] it is known that for a commutative ring A,

$$\pi_0(\mathrm{THH}(A)^{C_{p^n}}) \cong W_{n+1}(A),$$

where $W_{n+1}(A)$ denotes the p-typical Witt vectors of length $n+1$. In our work on algebraic models we prove

Theorem. *For a (-1)-connected commutative K-ring spectrum R and for $K \subset G \subset S^1$, we have a natural isomorphism*

$$\underline{\pi}_0^G(\mathrm{THH}^K(R)) \cong \underline{\mathrm{HH}}_0^G(\underline{\pi}_0^K R),$$

where THH^K *denotes the K-relative version of topological Hochschild homology, as in* [3].

Therefore, one can consider these equivariant Hochschild homology groups as a kind of Witt vectors for commutative Green functors.

In the ordinary topological Hochschild homology case, the relationship between THH and the Witt vectors extends to an algebraic structure on TR. In particular, Hesselholt-Madsen [41] showed that as n and q vary, for a ring A the groups $\mathrm{TR}_q^n(A;p)$ fit into a rigid algebraic structure called a Witt complex over A, which extends the relationship between THH (which is TR^1) and the Witt vectors. The category of Witt complexes over A has an initial object called the de Rham-Witt complex, and the relationship between the de Rham-Witt complex and the TR-theory of A can be very useful in understanding TR, and hence K-theory. This leads to some natural questions arising out of our work on algebraic models: How do we define the structure of a Witt complex over a commutative Green functor? How do we understand a de Rham-Witt complex over a commutative Green functor? These are a few of the questions that we are considering in ongoing work in this area.

2.4. **Topological coHochschild homology.**

Topological Hochschild homology is an analog of the classical Hochschild homology of algebras, in the context of ring spectra. There is also a classical theory of coHochschild homology for coalgebras [22], [17], [35]. In recent work, Hess and Shipley [36] defined a topological analog of this theory, called topological coHochschild homology (coTHH), for topological coalgebras. In recent joint work with Anna Marie Bohmann, Amalie Høgenhaven, Brooke Shipley, and Stephanie Zieganhagen [13], we developed computational tools for this new theory. For one, we proved a Hochschild–Kostant–Rosenberg type theorem for differential graded coalgebras in the cofree case:

Theorem. *Let Y be a nonnegatively graded cochain complex over a field k. Then*

$$\mathrm{coTHH}(S^c(Y)) \simeq S^c(Y) \otimes_k S^c(\Sigma^{-1}Y),$$

where $S^c(Y)$ be the cofree coaugmented cocommutative coassociative conilpotent coalgebra over k cogenerated by Y.

For ordinary topological Hochschild homology, an essential computational tool is the Bökstedt spectral sequence, where the E_2-term is expressed in terms of ordinary Hochschild homology. In our work, we look at the dual spectral sequence for coTHH, and show that the E_2-term there can be expressed in terms of classical coHochschild homology:

Theorem. *Let C be a coalgebra spectrum which is cofibrant. The Bousfield–Kan spectral sequence for $\mathrm{coTHH}(C)$ gives a coBökstedt spectral sequence with E_2-page*

$$E_2^{s,t} = \mathrm{coHH}_s(H_t(C; \mathbb{F}_p)),$$

that abuts to

$$H_{t-s}(\mathrm{coTHH}(C); \mathbb{F}_p).$$

Under some conditions, this spectral sequence converges strongly. Further, if C is a cocommutative coalgebra, this is a spectral sequence of coalgebras.

Using the coalgebra structure in the spectral sequence we are able to compute the homology of $\mathrm{coTHH}(C)$ for a number of coalgebra spectra C.

A number of interesting questions come out of this work. When X is simply connected we know that $\mathrm{coTHH}(\Sigma^\infty_+ X)$ receives a trace map from Waldhausen's $A(X)$. We would like to know in what generality we have trace maps connecting algebraic K-theory and topological coHochschlid homology. Further we would like to understand equivariant structures on coTHH. In particular, can we define a cyclotomic structure on $\mathrm{coTHH}(C)$ leading to a dual theory for TR and TC?

2.5. **Kaledin's Hodge-to-de Rham degeneration via homotopy theory.**

In [44] Kaledin proves a non-commutative Hodge-to-de Rham degeneration result. In particular, he proves:

Theorem. *For a saturated DG-algebra A^* over a field k of characteristic 0, the Hodge-to-de Rham spectral sequence*

$$\mathrm{HH}_*(A^*)[u^{-1}] \Rightarrow \mathrm{HC}_*(A^*)$$

degenerates at the first term, so that $\mathrm{HC}_*(A^*) \cong \mathrm{HH}_*(A^*)[u^{-1}]$.

This result partially addresses a conjecture of Kontsevich and Soibelman [45]. Kaledin states in his work that the motivation for his methods is coming from algebraic topology, but he uses purely homological tools. In an ongoing joint project with Jonathan Campbell, we are exposing the algebraic topology behind this result.

In particular, the crux of Kaledin's argument is producing an isomorphism

$$\mathrm{HH}_*(A^{*(1)})((u)) \cong \mathrm{HP}_*(A^*),$$

where A_* is a smooth DG-algebra over a finite field of characteristic $p > 0$, satisfying some additional hypotheses. Here HP denotes periodic homology, and the superscript (1) denotes a Frobenius twist. Kaledin studies these invariants using homological tools, but a topological perspective may yield more insight. Periodic homology can be identified as the homotopy groups of a Tate spectrum:

$$\pi_* \hat{\mathbb{H}}(S^1; \mathrm{HH}(A)) \cong \mathrm{HP}_*(A).$$

One can also identify $\mathrm{HH}_*(A)((u))$ as a version of periodic homology for p-cyclic objects

$$\mathrm{HH}_*(A)((u)) \cong \mathrm{HP}_*(\pi^* A),$$

where π is a projection functor from the p-cyclic category to the cyclic category. Thus $\mathrm{HH}_*(A)((u))$ can also be expressed as the homotopy groups of a Tate spectrum. We believe that Kaledin's isomorphism above can be given as a map of homotopy groups of Tate spectra. This involves using a splitting $HA \to HA \wedge Hk$ to get a map

$$\mathrm{HH}(A) = \mathrm{THH}_{Hk}(HA) \to \mathrm{THH}_{Hk}(HA \wedge Hk) = \mathrm{THH}(A) \wedge Hk,$$

then also using the cyclotomic structure on THH, and the linearization map. We are currently in the process of making our proof outline rigorous. This proof would not only illuminate the topological tools motivating Kaledin's result but it would demonstrate the power of algebraic topology to address questions in noncommutative geometry.

2.6. Defining an $RO(S^1)$-graded Witt Complex.

To understand the full equivariant homotopy theory of a G-spectrum, one should study homotopy groups graded by the real representation ring of G rather than the integers. It is these $RO(G)$-graded homotopy groups that appear in the computation of algebraic K-theory of singular varieties, and $RO(G)$-graded theories are also essential to the proof of the Kervaire Invariant one problem [42]. Despite these important applications, the structures that arise in these $RO(G)$-graded computations remain somewhat mysterious.

I am working to define a new algebraic structure, the $RO(S^1)$-*graded Witt complex*, which generalizes the ordinary Witt complex, and embodies the structure of $RO(S^1)$-graded TR. Understanding the algebraic structure of $RO(S^1)$-graded TR-groups would facilitate their computation, as well as giving a framework in which to express the results of such computations. In the non-equivariant case, a Witt complex $E^*_\bullet$ over a $\mathbb{Z}_{(p)}$-algebra A is a pro-differential graded ring with operators F and V satisfying certain relations. As shown in [40], the system of groups $\mathrm{TR}^\bullet_*(A;p)$ and their operators form a Witt complex over A. In the $RO(S^1)$-graded case it is immediate that the algebraic structure one sees is significantly different from the ordinary case. In ongoing work, I aim to define an $RO(S^1)$-graded Witt complex. The initial object in the category of $RO(S^1)$-graded Witt complexes over A would serve as an $RO(S^1)$-*graded de Rham-Witt complex of* A. An understanding of this universal object would lead to further computations of $RO(S^1)$-graded TR-groups, and hence algebraic K-theory groups.

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