Some remarks on the Smith-Pinio-McGehee proof of Littlewood's conjecture

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Abstract

The main goal of this paper is to show flexibility of the technique used by McGehee, Pinio and Smith for obtaining a lower bound for L^1-norm of trigonometric sums. Using the same method one can generalize the estimate onto the case of non-integer frequencies, obtain a new proof of the Bochkarev lemma on martingals and its analogue for trigonometric sums. Thus, one can see that the technique is not as tightly connected with exact orthogonality and "unidirectiveness" of spectrum as was thought before.

§0. Introduction.

This article has been written not for the concrete results it contains. Its main task is to show "flexibility" of the Smith-Pinio-McGehee technique [2], which most people consider to be very tightly connected with exact orthogonality and "unidirectiveness" of spectrum. Besides, he reader who knows the subject well will also note an essential simplification in the proof of the so called "difficult" Bochkarev lemma on martingals. Main cause of this simplification is that we use some automate (the function xe^{-x}) instead of a straightforward handiwork to split a function into levels and to choose maximal terms (It is this automatization that makes the idea of [2] so useful and attractive; in the case of Bochkarev's lemma it can be seen very clear and the reader may compare our proof with original one due to Bochkarev step by step to enjoy how neatly the function xe^{-x} works. I will only display in the article the original proof of the "simple" Bochkarev lemma for which the comparison is almost trivial but may be of some interest for the reader who has never dealt with the subject before).

In conclusion of this introduction I'd like to thank A.N.Podkorytov and S.V.Bochkarev who have acquainted me with this topic.

§1. Smith-Pinio-McGehee inequality for "well-separated" exponents.

It was conjectured by Littlewood that $\int_{-1/2}^{1/2}\left|\sum_{j=1}^{N} e^{2\pi i m_j t}\right| dt \geq c\log N$, where $m_1 < m_2 < \cdots \in \mathbb{Z}$ are arbitrary different integers. It was proved independently by Koniagin [1] and Smith, Pinio and McGehee [2]. In [2] essentially more general result was obtained:

$$\int_{-1/2}^{1/2}\left|\sum_{j=1}^{N} a_j e^{2\pi i m_j t}\right| dt \geq c\sum_{j=1}^{N}\frac{|a_j|}{j} \qquad (a_j \in \mathbb{C})$$

We are going to show that the same inequality is true for arbitrary exponential polynomial $p(t)=\sum_{j=1}^{N} a_j e^{2\pi i\lambda_j t}$ with real exponents $\lambda_j \in \mathbb{R}$ satisfying $\lambda_{j+1}-\lambda_j \geq 1$ if we replace the integral over $[-1/2, 1/2]$ by that over $I=[-\Delta/2, \Delta/2]$ with any $\Delta > 1$, the constant c in the right hand part depending on Δ only (unfortunately I don't know whether it is possible to take $\Delta = 1$). Our proof will differ from that in [2] very slightly. For convenience we denote $\delta = \Delta - 1$ and suppose indices of the exponents λ_j to change not from

1 to N but from $J_\delta + 1$ to $J_\delta + N$ where J_δ is some large positive integer depending on δ only (its precise value will be chosen later).

Let ψ be a C^∞-smooth function supported at the interval $(-\delta/2, \delta/2)$ and such that $\int_{\mathbb{R}} \psi = 1$, $\eta(t) = \begin{cases} \cos \pi t, & t \in (-\frac{1}{2}, \frac{1}{2}) \\ 0, & t \notin (-\frac{1}{2}, \frac{1}{2}) \end{cases}$, $\varphi = \frac{\pi}{2} \varphi * \psi$. Then $supp\, \varphi \subset I$, $\widehat{\varphi}(\lambda) = \frac{\cos \pi\lambda}{1-4\lambda^2} \widehat{\psi}(\lambda)$

If J_δ is large enough then $\sum_{\substack{j=J_\delta+1 \\ j\neq j_0}}^{J_\delta+N} \frac{|\widehat{\varphi}(\lambda_j - \lambda_{j_0})|}{j} \leq \frac{1-\eta}{j_0}$ for each $j_0 > J_\delta$ ($\eta > 0$ depends on δ only).

Indeed, $|\widehat{\psi}(\lambda)| \leq 1 - 3\eta$ when $|\lambda| > 1$ and therefore $|\widehat{\varphi}(\lambda)| \leq \frac{1-3\eta}{4\lambda^2 - 1}$ when $|\lambda| > 1$. So, $\sum_{j\neq j_0} |\widehat{\varphi}(\lambda_j - \lambda_{j_0})| \leq (1-3\eta) \sum_{j\neq j_0} \frac{1}{4(j-j_0)^2 - 1} = 1 - 3\eta$.

Beside it, since φ is smooth, $|\varphi(\lambda)| < \frac{1}{|\lambda|^3}$, provided that λ is greater than some positive constant $A > 0$. Write

$$\sum_{j\neq j_0} \frac{|\widehat{\varphi}(\lambda_j - \lambda_{j_0})|}{j} = \sum_{j<(1-\eta)j_0} + \sum_{j\geq(1-\eta)j_0} = \Sigma_1 + \Sigma_2.$$

The sum Σ_1 consists of at most j_0 terms. Each of them doesn't exceed $\frac{1}{(\eta j_0)^3}$ modulo, provided that $|\lambda_j - \lambda_{j_0}| \geq |j - j_0| \geq \eta j_0 \geq \eta J_\delta > A$ (This is a condition on J_δ). Thus

$$\Sigma_1 \leq j_0 \frac{1}{(\eta j_0)^3} \leq \frac{1}{\eta^3 J_\delta j_0} \leq \frac{\eta}{j_0}$$

for each $J_\delta > \max\{A\eta^{-1}, \eta^{-4}\}$.

From the other side

$$\Sigma_2 \leq \frac{1}{(1-\eta)j_0} \sum_{j\neq j_0} |\widehat{\varphi}(\lambda_j - \lambda_{j_0})| \leq \frac{1-3\eta}{1-\eta} \frac{1}{j_0} < \frac{1-2\eta}{j_0}.$$

Adding these inequalities we get the estimate desired.

Without loss of generality we may assume that J_δ is a power of 2, $J_\delta = 2^{m(\delta)}$, say. Let b_j be any complex numbers, equal to 1 modulo. For each $m \geq m(\delta)$ put

$$f_m(t) = \sum_{2^m < j \leq 2^{m+1}} \frac{b_j e^{-2\pi i \lambda_j t}}{j}.$$

So, for each $j_0 > J_\delta$ we have

$$\int_I \Big\{ \sum_{m \ge m(\delta)} f_m(t) \Big\} \varphi(t) e^{2\pi i \lambda_{j_0} t}\, dt = \frac{b_{j_0}}{j_0} + R_{j_0},$$

where $|R_{j_0}| = \left| \sum_{j \ne j_0} b_j \dfrac{\widehat{\varphi}(\lambda_j - \lambda_{j_0})}{j} \right| \le \sum_{j \ne j_0} \dfrac{|\widehat{\varphi}(\lambda_j - \lambda_{j_0})|}{j} \le \dfrac{1-\eta}{j_0}$

The well-known Salem inequality implies that $\|f_m\|_2 \le Const\, 2^{-m/2}$ (here $\|\cdot\|$ denotes the norm in $L^2(I)$). There exist functions $h_m(t) = \sum_{l=0}^{\infty} h_{m,l} e^{(2\pi/\Delta) ilt}$ (Δ-periodic) such that $\Re h_m = |f_m|$, $\|h_m\|_2 \le 2\|f_m\|_2 \le Const\, 2^{-m/2}$. Put $H_m(t) = \sum_{n \ge m} h_n(t)$. Obviously $\|H_m\|_2 \le Const\, 2^{-m/2}$ and $H_m(t) = \sum_{l=0}^{\infty} H_{m,l} e^{(2\pi/\Delta) ilt}$.

Fix small $\varepsilon > 0$ and put

$$F(t) = \sum_{m \ge m(\delta)} f_m(t) e^{-\varepsilon H_m(t)}.$$

It is easy to check that $\|F\|_\infty \le \dfrac{1}{\varepsilon}$. Let $2^{m_0} < j_0 \le 2^{m_0+1}$, $m_0 \ge m(\delta)$. Let's now calculate the integral

$$\int_I F(t) e^{2\pi i \lambda_{j_0} t} \varphi(t)\, dt = \int_I \Big\{ \sum_{m \ge m(\delta)} f_m(t) \Big\} e^{2\pi i \lambda_{j_0} t} \varphi(t)\, dt +$$
$$\int_I \Big\{ \sum_{m \ge m(\delta)} f_m(t) \big(e^{-\varepsilon H_m(t)} - 1 \big) \Big\} e^{2\pi i \lambda_{j_0} t} \varphi(t)\, dt = \mathcal{J}_1 + \mathcal{J}_2$$

$\mathcal{J}_1$ has already been estimated above:

$$\mathcal{J}_1 = \frac{b_{j_0}}{j_0} + R_{j_0} \qquad |R_{j_0}| \le \frac{1-\eta}{j_0}$$

Further, we have

$$\mathcal{J}_2 = \int_I \sum_{m \le m_0 - 2} \cdots + \int_I \sum_{m \ge m_0 - 1} \cdots = \mathcal{J}_2' + \mathcal{J}_2''$$

We shall estimate $\mathcal{J}_2''$ in the most trivial way:

$$|\mathcal{J}_2''| \le Const \sum_{m \ge m_0 - 1} \varepsilon \|f_m\|_2 \|H_m\|_2 \le Const\, \varepsilon \sum_{m \ge m_0 - 1} 2^{-m} =$$
$$Const\, 2^{-m_0} \le \frac{Const\, \varepsilon}{j_0} < \frac{\eta}{3 j_0},$$

provided that ε is small enough.

The function $e^{-\varepsilon H_m(t)} - 1$ can be represented as $e^{-\varepsilon H_m(t)} - 1 = \sum_{l=0}^{\infty} q_{m,l} e^{(2\pi/\Delta)ilt}$, all the coefficients $q_{m,l}$ being less than 2 modulo (for the function doesn't exceed 2 modulo at each point). Therefore the sum in the definition of $\mathcal{J}_2'$ consists of at most j_0 terms of the kind

$$\frac{b_j}{j} e^{-2\pi i\lambda_j t} \left(\sum_{l=0}^{\infty} q_l e^{(2\pi/\Delta)ilt} \right) e^{2\pi i\lambda_{j_0} t} \varphi(t)$$

where $j \leq \frac{j_0}{2}$, $|q_l| \leq 2$.

The integral of each of these terms over I can be estimated from above by

$$2 \sum_{l=0}^{\infty} |\widehat{\varphi}(\lambda_j - \lambda_{j_0} - \frac{2\pi}{\Delta} l)| \leq \frac{\eta}{12(\lambda_{j_0} - \lambda_j)^2},$$

provided that $\lambda_{j_0} - \lambda_j$ is large enough (we use here the smoothness of φ again). But $\lambda_{j_0} - \lambda_j \geq \frac{j_0}{2} \geq \frac{J_\delta}{2}$, and again it is enough to choose J_δ not very small. Thus,

$$|\mathcal{J}_2'| \leq j_0 \frac{\eta}{12(j_0/2)^2} = \frac{\eta}{3j_0}$$

Finally we have:

$$\int_I F(t) e^{2\pi i\lambda_{j_0} t} \varphi(t)\, dt = \frac{b_{j_0}}{j_0} + \widetilde{R}_{j_0} \ , \quad |\widetilde{R}_{j_0}| \leq \frac{1 - \eta/3}{j_0}.$$

The rest doesn't differ a bit from the classic reasoning for integer exponents: Putting $b_j = \overline{sign\, a_j}$, we find:

$$\left| \int_I \left(\sum_j a_j e^{2\pi i\lambda_j t} \right) F(t)\varphi(t)\, dt \right| = \left| \sum_j \left(\frac{|a_j|}{j} + a_j \widetilde{R}_j \right) \right| \geq \frac{\eta}{3} \sum_{j=J_\delta+1}^{J_\delta+N} \frac{|a_j|}{j},$$

and since $|F(t)\varphi(t)| \leq \frac{\pi}{2\varepsilon}$ we obtain

$$\left\| \sum_{j=J_\delta+1}^{J_\delta+N} a_j e^{2\pi i\lambda_j t} \right\|_1 \geq \frac{2\varepsilon\eta}{3\pi} \sum_{j=J_\delta+1}^{J_\delta+N} \frac{|a_j|}{j}.$$

If we return to the initial enumeration, we shall get

$$\left\| \sum_{j=1}^{N} a_j e^{2\pi i\lambda_j t} \right\|_1 \geq \frac{2\varepsilon\eta}{3\pi} \sum_{j=1}^{N} \frac{|a_j|}{j + J_\delta} \geq \frac{2\varepsilon\eta}{3\pi(J_\delta + 1)} \sum_{j=1}^{N} \frac{|a_j|}{j},$$

which completes the proof.

§2. The Bochkarev lemma for marthingals

In 1974 S.V.Bochkarev proved the following inequality

The "simple" Bochkarev lemma

Let χ_k be the Haar functions, i.e. $\chi_k : [0,1] \to \mathbb{R}$,

$$\chi_k(t) = \begin{cases} 0\,, \quad t \notin [\frac{k-2^s-1}{2^s}, \frac{k-2^s}{2^s}) \\ 2^{s/2}, \, t \in [\frac{k-2^s-1}{2^s}, \frac{k-2^s-1/2}{2^s}) \\ -2^{s/2}, \, t \in [\frac{k-2^s-1/2}{2^s}, \frac{k-2^s}{2^s}) \end{cases} \qquad (2^s < k \le 2^{s+1}; \quad s = 0,1,2,\ldots)$$

Let $\delta_s = \sum\limits_{2^{s-1}<k\le 2^s} c_k\chi_k$. The functions δ_s are characterized by the following two properties: δ_s is constant on each diadic interval of length 2^{-s} and its integral over each diadic interval of length $2^{-(s-1)}$ equals 0. By other words, the sequence of partial sums of the series $\sum_{s=0}^{\infty}\delta_s$ forms a marthingal with respect to the sequence of diadic σ-algebras (only this marthingal property will be used below; so, the following result remains valid for arbitrary marthingals and belongs rather to probability theory than to analysis).

Let q_s be a sequence of positive real numbers satisfying $q_{s+1} \le (1-\sigma)q_s$ for each $s = 1,2,\ldots$ with some $\sigma > 0$. If $\|\delta_s\|_\infty \le q_s$ for each s then

$$\int_0^1 \sum_{s=1}^{\infty} \frac{1}{q_s}\delta_s^2(x)\,dx \le \frac{C}{\sigma}\int_0^1 \left|\sum_{s=1}^{\infty}\delta_s(x)\right| dx.$$

There is a generalization of the last inequality also due to Bochkarev:

The "difficult" Bochkarev lemma.

Let δ_s and q_s be the same as above, $0 = s_0 < s_1 < s_2 < \cdots \in \mathbb{Z}_+$ be an increasing sequence of positive integers. Suppose that instead of $||\delta_s||_\infty \le q_s$ we know that

$$\left\|\left(\sum_{s_{m-1}<s\le s_m}\delta_s^2\right)^{1/2}\right\|_\infty \le q_m.$$

Then

$$\int_0^1 \left\{\sum_{m=1}^{\infty}\frac{1}{q_m}\left(\sum_{s_{m-1}<s\le s_m}\delta_s^2(x)\right)\right\} dx \le \frac{C}{\sigma}\int_0^1\left|\sum_{s=1}^{\infty}\delta_s(x)\right| dx.$$

Bochkarev applied these lemmas to construct a L^1-function Fourier series of which with respect to an arbitrary uniformly bounded orthogonal system diverges on a set of positive measure and a function of bounded variation with prescribed module of continuity ω satisfying $\sum\limits_{n=1}^{\infty}\frac{\sqrt{\omega(1/n)}}{n} = \infty$, Fourier series of which with respect to an arbitrary uniformly bounded orthogonal system doesn't converge absolutely, correspondingly.

Proof of the "simple" Bochkarev lemma.

As promised, we will give two proofs.

Proof I (Bochkarev, 1974)

Define sets $E_{m,s}$ as follows:

$$E_{m,s} = \left\{ \frac{q_s}{2^{m+1}} < |\delta_s(x)| \le \frac{q_s}{2^m} \right\}, \qquad m = 0, 1, 2, \dots; s = 1, 2, 3, \dots.$$

We have:

$$\int_0^1 \sum_{s=1}^{\infty} \frac{1}{q_s} \delta_s^2(x)\, dx = \sum_{m=0}^{\infty} \sum_{s=1}^{\infty} \frac{1}{q_s} \int\limits_{E_{m,s}} \delta_s^2(x)\, dx.$$

For fixed m the sets $E_{m,s}$ with different s may intersect. To avoid this inconvenience put $H_{m,s} = E_{m,s} \setminus \bigcup_{r=0}^{s-1} E_{m,r}$

Since $H_{m,r}$ can intersect $E_{m,s}$ only when $r \le s$ we get

$$\int_0^1 \sum_{s=1}^{\infty} \frac{1}{q_s} \delta_s^2(x) dx = \sum_{m=0}^{\infty} \sum_{s=1}^{\infty} \frac{1}{q_s} \sum_{r=1}^{s} \int\limits_{E_{m,s} \cap H_{m,r}} \delta_s^2(x) dx = \sum_{m=0}^{\infty} \sum_{r=1}^{\infty} \sum_{s=r}^{\infty} \frac{1}{q_s} \int\limits_{E_{m,s} \cap H_{m,r}} \delta_s^2(x) dx$$

It follows immediately from the definitions of $E_{m,s}$ and $H_{m,r}$ that

$$\int\limits_{E_{m,s} \cap H_{m,r}} \delta_s^2(x)\, dx \le \frac{q_s^2}{2^{2m}} mes\, H_{m,r}$$

$$\frac{q_r}{2^{m+1}} mes\, H_{m,r} \le \int\limits_{H_{m,r}} |\delta_r(x)|\, dx$$

Using these inequalities and the obvious estimate $\sum\limits_{s=r}^{\infty} q_s \le \frac{1}{\sigma} q_r$, we obtain

$$\int_0^1 \sum_{s=1}^{\infty} \frac{1}{q_s} \delta_s^2(x)\, dx \le \sum_{m=0}^{\infty} \frac{1}{2^{2m}} \sum_{r=1}^{\infty} mes H_{m,r} \sum_{s=r}^{\infty} q_s \le \frac{2}{\sigma} \sum_{m=0}^{\infty} \frac{1}{2^m} \sum_{r=1}^{\infty} \int\limits_{H_{m,r}} |\delta_r(x)|\, dx$$

Note that the sets $E_{m,r}$ consist of entire diadic intervals the Haar functions of s-th bundle δ_s are supported on. So, the sets $H_{m,r}$ possess the same property and therefore

$$\int\limits_{H_{m,r}} |\delta_r(x)|\, dx \le \int\limits_{H_{m,r}} |\sum_{s=1}^{\infty} \delta_s(x)|\, dx$$

Using the last two inequalities and taking into account that the sets $H_{m,s}$ are mutually disjoint for fixed m , we get

$$\int_0^1 \sum_{s=1}^{\infty} \frac{1}{q_s}\delta_s^2(x)\,dx \le \frac{4}{\sigma}\int_0^1 |\sum_{s=1}^{\infty}\delta_s(x)|\,dx$$

which completes the proof.
Proof II (compare with [2])

Fix $\varepsilon > 0$ and put

$$F = \sum_{s=1}^{\infty} \frac{\delta_s}{q_s}\exp\Big\{-\varepsilon\sum_{r=1}^{s}\frac{|\delta_r|}{q_r}\Big\}.$$

Note that $\|F\|_\infty \le \dfrac{1}{\varepsilon}$ and for each $p \in \mathbb{N}$

$$\int_0^1 F\delta_p = \int_0^1 \frac{\delta_p^2}{q_p}\exp\Big\{-\varepsilon\sum_{r=1}^{p}\frac{|\delta_r|}{q_r}\Big\}.$$

The latter follows from the simple observation that the integral $\displaystyle\int_0^1 \frac{\delta_s}{q_s}\exp\Big\{-\varepsilon\sum_{r=1}^{s}\frac{|\delta_r|}{q_r}\Big\}\delta_p$ equals 0 if $p > s$ because first two factors are constant on diadic intervals of length 2^{-s} while the integral of δ_p over each such an interval equals 0. If, conversely, $p > s$ then the last two factors are constant on diadic intervals of length 2^{s-1} while the integral of δ_s over each such an interval equals 0 (The fact that $|\delta_s|$ is constant on twice longer intervals than δ_s is, of course, nothing but a lucky coincidence but it is too pleasant to disregard it. When we prove the more general "difficult" Bochkarev lemma we shall not use this specific feature of the Haar system).

Thus

$$\int_0^1 F\sum_{s=1}^{\infty}\delta_s = \sum_{s=1}^{\infty}\int_0^1 \frac{\delta_s^2}{q_s}\exp\Big\{-\varepsilon\sum_{r=1}^{s}\frac{|\delta_r|}{q_r}\Big\} \ge \sum_{s=1}^{\infty}\int_0^1\frac{\delta_s^2}{q_s} - \varepsilon\sum\sum_{r,s:\ r\le s}\int_0^1 \frac{\delta_s^2|\delta_r|}{q_s q_r}$$

(we have used the inequality $e^{-\varepsilon t} \ge 1 - \varepsilon t$)

But

$$\sum\sum_{r,s:\ r\le s}\int_0^1 \frac{\delta_s^2|\delta_r|}{q_s q_r} \overset{(|\delta_s|\le q_s)}{\le} \sum\sum_{r,s:\ r\le s}\int_0^1\frac{|\delta_s||\delta_r|}{q_r} \le \sum\sum_{r,s:\ r\le s}\int_0^1 \frac{1}{2}\sqrt{\frac{q_s}{q_r}}\left(\frac{\delta_s^2}{q_s}+\frac{\delta_r^2}{q_r}\right)$$

$$= \frac{1}{2}\Big\{\sum\sum_{r,s:\ r\le s}\sqrt{\frac{q_s}{q_r}}\int_0^1\frac{\delta_s^2}{q_s} + \sum\sum_{r,s:\ r\le s}\sqrt{\frac{q_s}{q_r}}\int_0^1\frac{\delta_r^2}{q_r}\Big\} \le \frac{2}{\sigma}\sum_{s=1}^{\infty}\int_0^1\frac{\delta_s^2}{q_s},$$

because it follows from the condition $q_{j+1} \le (1-\sigma)q_j$ that for each $s \in \mathbb{N}$ $\displaystyle\sum_{r=1}^{s}\sqrt{\frac{q_s}{q_r}} \le \frac{1}{1-\sqrt{1-\sigma}} \le \frac{2}{\sigma}$ and for each $r \in \mathbb{N}$ $\displaystyle\sum_{s=r}^{\infty}\sqrt{\frac{q_s}{q_r}} \le \frac{2}{\sigma}.$

Therefore, if we put $\varepsilon = \frac{\sigma}{4}$, we shall get

$$\int_0^1 F \sum_{s=1}^{\infty} \delta_s \geq \frac{1}{2} \sum_{s=1}^{\infty} \int_0^1 \frac{\delta_s^2}{q_s}$$

and taking into account that $\|F\|_\infty \leq \frac{4}{\sigma}$

$$\sum_{s=1}^{\infty} \int_0^1 \frac{\delta_s^2}{q_s} \leq \frac{8}{\sigma} \int_0^1 \left| \sum_{s=1}^{\infty} \delta_s \right|$$

Let's now prove the "difficult" Bochkarev lemma.

Put $f_m = \sum_{s_{m-1}<s\leq s_m} \delta_s$. Note that the functions f_m are constant on diadic intervals of length 2^{s_m} and have zero integral over each diadic interval of length $2^{s_{m-1}}$. To repeat the proof of the "simple" case word by word we lack only one thing: the inequality $|f_m| \leq q_m$. Now we are going to change functions f_m a little to satisfy this condition.

Lemma.

There exist functions g_m such that

(1) $\|g\|_\infty \leq 2q_m, \qquad \|g_m\|_2 \leq \|f_m\|_2$

(2) $\int_0^1 f_m g_m \geq \frac{1}{2} \int_0^1 f_m^2$

(3) g_m is constant on each diadic interval of length 2^{-s_m} and has zero integral over each diadic interval of length $2^{-s_{m-1}}$.

Suppose the lemma proved. Fix small $\varepsilon > 0$ and put

$$F = \sum_{m=1}^{\infty} \frac{g_m}{q_m} \exp\left\{ -\varepsilon \sum_{r=1}^{m-1} \frac{|g_r|}{q_r} \right\}.$$

Note that $\|F\|_\infty \leq \frac{e^{2\varepsilon}}{\varepsilon} \leq \frac{e^2}{\varepsilon}$.

Property (3) implies

$$\begin{aligned}\int_0^1 F \sum_{m=1}^{\infty} f_m &= \sum_{m=1}^{\infty} \int_0^1 \frac{g_m}{q_m} \exp\left\{ -\varepsilon \sum_{r=1}^{m-1} \frac{|g_r|}{q_r} \right\} f_m \\ &\geq \sum_{m=1}^{\infty} \int_0^1 \frac{g_m f_m}{q_m} - \varepsilon \sum_{r,m:\, r<m} \int_0^1 \frac{|g_m||f_m||g_r|}{q_m q_r}\end{aligned}$$

But $\int_0^1 \frac{g_m f_m}{q_m} \geq \frac{1}{2} \int_0^1 \frac{f_m^2}{q_m}$, while

$$\sum_{r,m:\,r<m}\sum\int_0^1 \frac{|g_m|\,|f_m|\,|g_r|}{q_m q_r} \overset{(|g_m|\le q_m)}{\le} 2\sum_{r,m:\,r<m}\sum\int_0^1 \frac{|f_m|\,|g_r|}{q_r}$$

$$\le \sum_{r,m:\,r<m}\sum\int_0^1 \sqrt{\frac{q_m}{q_r}}\Big(\int_0^1 \frac{f_m^2}{q_m}+\int_0^1 \frac{g_r^2}{q_r}\Big) \le \frac{4}{\sigma}\sum_{m=1}^{\infty}\int_0^1 \frac{f_m^2}{q_m}$$

Putting here $\varepsilon=\dfrac{\sigma}{16}$, we get

$$\int_0^1\Big\{\sum_{m=1}^{\infty}\frac{1}{q_m}\Big(\sum_{s=s_{m-1}+1}^{s_m}\delta_s^2(x)\Big)\Big\}\,dx=\sum_{m=1}^{\infty}\int_0^1\frac{f_m^2}{q_m}\le$$

$$\frac{64}{\sigma}\int_0^1\Big|\sum_{m=1}^{\infty}f_m(x)\Big|\,dx=\frac{64}{\sigma}\int_0^1\Big|\sum_{s=1}^{\infty}\delta_s(x)\Big|\,dx$$

because it follows from the condition $q_{j+1}\le(1-\sigma)q_j$ that for each $s\in\mathbb{N}$ $\displaystyle\sum_{r=1}^{s}\sqrt{\frac{q_s}{q_r}}\le\frac{1}{1-\sqrt{1-\sigma}}\le\frac{2}{\sigma}$ and for each $r\in\mathbb{N}$ $\displaystyle\sum_{s=r}^{\infty}\sqrt{\frac{q_s}{q_r}}\le\frac{2}{\sigma}$.

Therefore, putting $\varepsilon=\dfrac{\sigma}{4}$, we obtain

$$\int_0^1 F\sum_{s=1}^{\infty}\delta_s\ge\frac{1}{2}\sum_{s=1}^{\infty}\int_0^1\frac{\delta_s^2}{q_s}$$

and taking into account that $\|F\|_\infty\le\dfrac{4}{\sigma}$,

$$\sum_{s=1}^{\infty}\int_0^1\frac{\delta_s^2}{q_s}\le\frac{8}{\sigma}\int_0^1\Big|\sum_{s=1}^{\infty}\delta_s\Big|$$

It remains to construct functions g_m with properties desired. It may be done in many ways, all these ways being quite standard. We shall choose the most direct of them: the stop moments technique.

Let $f_m^{(j)}\overset{def}{=}\displaystyle\sum_{s=s_{m-1}+1}^{j}\delta_s\qquad(j=s_{m-1}+1,\dots,s_m)$

Put $E_{s_{m-1}}=[0,1], E_s=\{|f_m^{(j)}|\le q_m\quad s_{m-1}<j\le s\}\qquad(s_{m-1}<s\le s_m)$

Denote $g_m=\displaystyle\sum_{s_{m-1}<s\le s_m}\chi_{E_{s-1}}\delta_s$ (by other words, we stop to add terms into f_m in the very moment the sum becomes greater than q_m modulo). Obviously, $\|g_m\|_\infty\le 2q_m$ and g_m is constant on each diadic interval of length 2^{-s_m}. Beside it, as E_{s-1} consists of entire diadic intervals of length $2^{-(s-1)}$ and the integral of δ_s over every such interval equals 0,

the integral of g_m over each diadic interval of length $2^{-s_{m-1}}$ equals 0. Further, on the set E_{s_m-1} we have $g_m \equiv f_m$ and therefore

$$\int\limits_{E_{s_m-1}} g_m^2 = \int\limits_{E_{s_m-1}} g_m f_m = \int\limits_{E_{s_m-1}} f_m^2.$$

For each $s_{m-1} < s < s_m$ on $E_{s-1} \setminus E_s$ we have: $f_m = g_m + \sum\limits_{s<r\le s_m} \delta_r$, functions g_m and δ_r $(s < r \le s_m)$ being orthogonal on $E_{s-1} \setminus E_s$ ($E_{s-1} \setminus E_s$ consists of entire diadic intervals of length 2^{-s}, the function $g_m = f_m^{(s)}$ is constant on each of them while the function δ_r with $s < r \le s_m$ has zero integral over each of them. The mutual orthogonality of the functions δ_r can be obtained in the same way)

Thus,

$$\int\limits_{E_{s-1}\setminus E_s} f_m^2 = \int\limits_{E_{s-1}\setminus E_s} g_m^2 + \int\limits_{E_{s-1}\setminus E_s} \sum_{s<r\le s_m} \delta_r^2.$$

But $g_m^2 \ge q_m^2$ on $E_{s-1} \setminus E_s$ while $\sum\limits_{s<r\le s_m} \delta_r^2 \le \sum\limits_{s_{m-1}<r\le s_m} \delta_r^2 \le q_m^2$ and therefore

$$\int\limits_{E_{s-1}\setminus E_s} f_m g_m = \int\limits_{E_{s-1}\setminus E_s} g_m^2 \ge \frac{1}{2} \int\limits_{E_{s-1}\setminus E_s} f_m^2.$$

Uniting these estimates, we find

$$\int_0^1 f_m g_m \ge \frac{1}{2} \int_0^1 f_m^2.$$

The lemma is proved.

§3. The Bochkarev lemma for the trigonometric system.

We shall consider the “difficult” case immediately:

Proposition.

Let $\delta_s(z) = \sum\limits_{2^{s-1}<|k|\le 2^s} c_k z^k$; $\quad 0 = s_0 < s_1 < \cdots \in \mathbb{Z}_+$; $\quad \{q_m\}$ be a sequence of positive numbers satisfying $q_{m+1} \le (1-\sigma) q_m$. Suppose that $\left\| \Big(\sum\limits_{s_{m-1}<s\le s_m} \delta_s^2 \Big)^{1/2} \right\|_\infty \le q_m$.

Then

$$\int\limits_{\mathbb{T}} \left\{ \sum_{m=1}^{\infty} \frac{1}{q_m} \Big(\sum_{s_{m-1}<s\le s_m} |\delta_s(z)|^2 \Big) \right\} d\mu(z) \le \frac{C}{\sigma} \int\limits_{\mathbb{T}} \left| \sum_{s=1}^{\infty} \delta_s(x) \right| d\mu$$

(Here μ denotes Lebesgue measure on the unit circumference $\mathbb{T}$ normalized by $\mu(\mathbb{T}) = 1$).

Proof.

As in the marthingal case, the proof consists of 2 parts: the construction of the auxiliary functions g_m and the application of the Smith technique itself. As above, put

$f_m = \sum_{s_{m-1}<s\le s_m} \delta_s$. Note that the spectrum of f_m is contained in the "symmetrical interval" $I(2^{s_{m-1}}, 2^{s_m})$ (for convenience we denote by $I(a,b)$ the union $[-b,-a)\cup(a,b]$)

Construction of the auxiliary functions g_m.

We need to construct functions g_m such that

$$\begin{array}{ll} (1) & \|g\|_\infty \le Const\, q_m, \qquad \|g_m\|_2 \le \|f_m\|_2 \\ (2) & \Re \int_0^1 f_m g_m \ge const \int_0^1 |f_m|^2 \\ (3) & spec\, g_m \subset I(2^{s_{m-1}}, 2^{s_m}) \end{array}$$

Split the function f_m into two terms: $f_m = (\delta_{s_{m-1}+1} + \delta_{s_m}) + \sum_{s=s_{m-1}+2}^{s_m-1} \delta_s = f'_m + f''_m$. The functions f'_m and f''_m are orthogonal and $\int_{\mathbb{T}} \overline{f}'_m f_m + \int_{\mathbb{T}} \overline{f}''_m f_m = \|f_m\|_2^2$.

Two cases are possible:

1) $\Re \int_{\mathbb{T}} \overline{f}'_m f_m \ge \frac{1}{2}\|f_m\|_2^2$.

In this case we may take $g_m = \overline{f}'_m$

2) $\Re \int_{\mathbb{T}} \overline{f}''_m f_m \ge \frac{1}{2}\|f_m\|_2^2$.

Here the construction is a bit more complicated. Put $h_m = \overline{f}''_m \exp\{-\eta \frac{|f''_m|}{q_m}\}$ ($\eta > 0$ is a small positive number to be chosen later). We have

$$\|h_m - \overline{f}''_m\|_2 \le \|\overline{f}''_m(\exp\{-\eta\frac{|f''_m|}{q_m}\} - 1)\|_2 \le \eta\|\overline{f}''_m \frac{|f''_m|}{q_m}\|_2 = \frac{\eta}{q_m}\|f''_m\|_4^2$$

According to the Littlewood-Paley inequality,

$$\|f''_m\|_p^p \le C_p \|f''_m\|_2^2 \left\| \left(\sum_{s=s_{m-1}+2}^{s_m-1} |\delta_s|^2 \right)^{1/2} \right\|_\infty^{p-2}$$

for each $p > 2$.

In particular $\|f''_m\|_4^4 \le C_4 \|f''_m\|_2^2 q_m^2$ and therefore $\|f''_m\|_4^2 \le Const\, q_m \|f''_m\|_2$.

Thus, if η is small enough

$$\|h_m - \overline{f}''_m\|_2 \le \frac{1}{24}\|f''_m\|_2 \le \frac{1}{24}\|f_m\|_2$$

while, obviously, $\|h_m\|_\infty \le \frac{1}{\eta} q_m$. The function h_m is almost that we need. It remains only to correct its spectrum.

Recall that the Valle-Poussin kernel K_j is the difference of two Fejer kernels $K_j = 2\Phi_{2j} - \Phi_j$. The convolution with the kernel K_j preserves the Fourier coefficients with indices less than or equal to j modulo and kills those with indices greater than $2j$ modulo. Beside it, the norm of the corresponding convolution operator in each space $L^p(\mathbb{T})$ $(1 \le p \le \infty)$ doesn't exceed 3.

Let Q_m be a convolution operator with the kernel $K_{2^{s_m-1}} - K_{2^{s_m-1}}$. Put $g_m = Q_m h_m$. It's easy to check that $spec\, g_m \subset I(2^{s_{m-1}}, 2^{s_m})$, $\|g_m\|_\infty \le 6\|h_m\|_\infty \le \dfrac{6}{\eta} q_m$ and $\|g_m\|_2 \le \|h_m\|_2 \le \|f_m\|_2$.

It remains only to note that

$$\Re \int\limits_{\mathbb{T}} g_m f_m = \Re \int\limits_{\mathbb{T}} (Q_m h_m) f_m = \Re \int\limits_{\mathbb{T}} (Q_m \overline{f}''_m) f_m + \Re \int\limits_{\mathbb{T}} Q_m (h_m - \overline{f}''_m) f_m \ge$$
$$\Re \int\limits_{\mathbb{T}} \overline{f}''_m f_m - 6\|h_m - \overline{f}''_m\|_2 \|f_m\|_2 \ge \frac{1}{2}\|f_m\|_2^2 - \frac{1}{4}\|f_m\|_2^2 = \frac{1}{4}\|f_m\|_2^2 ,$$

for $Q_m \overline{f}''_m = \overline{f}''_m$.

The construction of auxiliary functions g_m is now complete.

To proceed further we need the following simple

Lemma.

Let $g(z)$ be a trigonometric polynomial spectred at $[-4M, 4M]$ $(M \in \mathbb{N})$. Then there exists a nonnegative trigonometric polynomial $p(z)$ spectred at $[-M, M]$ and such that

$$\begin{array}{ll} (1) & const\,|g(z)| \le p(z) \le \|g\|_\infty \quad \text{for each } z \in \mathbb{T} \\ (2) & \|p\|_2 \le Const\,\|g\|_2 \end{array}$$

(all the constants are absolute)

Proof.

Split the unit circumference $\mathbb{T}$ into M equal arcs $\mathcal{L}_i$ and put $\varphi(z) = \max\limits_{\mathcal{L}_i} |g|$ for $z \in \mathcal{L}_i$. It is easy to derive from the S.N.Bernstein inequality $\|\frac{d}{dz} g\|_2 \le 4M\|g\|_2$ that $\|\varphi\|_2 \le Const\,\|g\|_2$. Beside it, obviously $\|\varphi\|_\infty \le \|g\|_\infty$.

Put $p = \varphi * \Phi_M$ (Φ_M denotes here the Fejer kernel). It is clear that $\|p\|_\infty \le \|\varphi\|_\infty \le \|g\|_\infty$, $\|p\|_2 \le \|\varphi\|_2 \le Const\,\|g\|_2$. The inequality $p(z) \ge const\,|g(z)|$ follows immediately from the observation that for each $z \in \mathcal{L}_i$

$$p(z) = \int\limits_{\mathbb{T}} \varphi(\zeta)\Phi_M(z\zeta^{-1})\, d\mu(\zeta) \ge \int\limits_{\mathcal{L}_i} \varphi(\zeta)\Phi_M(z\zeta^{-1}) d\mu(\zeta) = \max_{\mathcal{L}_i} \int\limits_{z\mathcal{L}_i^{-1}} \Phi_M\, d\mu \ge const\,|g(z)|$$

(the letter results from the evident estimate $\max\limits_{\mathcal{L}_i} |g| \ge |g(z)|$ and the fact that a large part of the Fejer kernel Φ_M of order M is concentrated on the arc $z\mathcal{L}_i^{-1}$ of length $\frac{2\pi}{M}$ containing the point 1.)

Lemma is proved.

Applying the lemma to each of the trigonometric polynomials g_m we get a sequence of nonnegative trigonometric polynomials $\{p_m\}$ such that

$$\begin{array}{ll} (1) & \|p_m\|_\infty \le Const\, q_m \\ (2) & \|p_m\|_2 \le Const\, \|g_m\|_2 \\ (3) & p_m \ge const\, |g_m| \\ (4) & spec\, p_m \subset [-2^{s_m-2}, 2^{s_m-2}] \end{array}$$

Fix small $\varepsilon > 0$ and put

$$F = \sum_{m=1}^{\infty} \frac{g_m}{q_m} \prod_{r=1}^{m-1} \left(1 - \frac{\varepsilon p_r}{q_r}\right).$$

If ε is small enough, the norm $\left\| \frac{\varepsilon p_r}{q_r} \right\|_\infty < 1$ and thereby

$$\prod_{r=1}^{m-1} \left(1 - \frac{\varepsilon p_r}{q_r}\right) \le \exp\left\{-\varepsilon \sum_{r=1}^{m-1} \frac{p_r}{q_r}\right\} \le \exp\left\{-const\, \varepsilon \sum_{r=1}^{m-1} \frac{|g_r|}{q_r}\right\}$$

So, we have $\|F\|_\infty \le \dfrac{e^2}{const\, \varepsilon}$ again.

Note now that

$$spec \prod_{r=1}^{m-1} \left(1 - \frac{\varepsilon p_r}{q_r}\right) \subset \left[-\sum_{r=1}^{m-1} 2^{s_r-2}, \sum_{r=1}^{m-1} 2^{s_r-2}\right] \subset \left(-2^{s_{m-1}-1}, 2^{s_{m-1}-1}\right)$$

We derive immediately from here that the spectrum

$$spec \left\{\frac{g_m}{q_m} \prod_{r=1}^{m-1} \left(1 - \frac{\varepsilon p_r}{q_r}\right)\right\} \subset I(2^{s_{m-1}} - 2^{s_{m-1}-1}, 2^{s_m} + 2^{s_{m-1}-1}) \subset I(2^{s_{m-2}}, 2^{s_{m+1}})$$

and thereby doesn't intersect the spectrum of f_l when $|l-m| > 1$.

Thus,

$$\int_{\mathbb{T}} F \sum_{m=1}^{\infty} f_m \, d\mu = \sum_{m=1}^{\infty} \int_{\mathbb{T}} \frac{g_m}{q_m} \prod_{r=1}^{m-1} \left(1 - \frac{\varepsilon p_r}{q_r}\right) (f_{m-1} + f_m + f_{m+1}) \, d\mu.$$

The rest is clear.

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