LOCAL ESTIMATES OF EXPONENTIAL POLYNOMIALS AND THEIR APPLICATIONS TO INEQUALITIES OF UNCERTAINTY PRINCIPLE TYPE

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Abstract. The paper is devoted to certain inequalities (modelled on the Turan lemma) for exponential polynomials, and to applications of these inequalities to various uniqueness theorems in harmonic analysis, like the uncertainty principle. An estimate is obtained of the maximum of the absolute value of an exponential polynomial

$$p(t) = \sum_{k=1}^{n} c_k e^{\lambda_k t}$$

($c_k \in \mathbb{C}$, $\lambda_k \in \mathbb{C}$) in terms of the maximum of its absolute value on a measurable set $E \subset I$ of positive Lebesgue measure:

$$\sup_{t\in I} |p(t)| \leqslant e^{\max|\operatorname{Re}\lambda_k|\mu(I)} \left\{ \frac{A\mu(I)}{\mu(E)} \right\}^{n-1} \sup_{t\in E} |p(t)|.$$

As for applications, the inequality

$$\|f\|^2_{L^2(\mathbb{R})} \leqslant Ae^{A\mu(E)\mu(\Sigma)} \left(\int_{\mathbb{R}\setminus E} |f|^2 + \int_{\mathbb{R}\setminus\Sigma} |\widehat{f}|^2 \right)$$

is proved (here f is an arbitrary function from $L^2(\mathbb{R})$ and E, Σ are arbitrary sets of finite Lebesgue measure), as well as a theorem on the integrability of small powers (less than $1/2$) of the logarithm of the absolute value of a function $f \in L^2(\mathbb{T})$ with lacunary spectrum in the sense of Zygmund.

Introduction

Chapter 1. Estimates of exponential polynomials

§1.1. Historical remarks

Key words and phrases. Exponential polynomial, uncertainty principle in harmonic analysis, lacunary trigonometric series.

Introduction

Exponential polynomials constitute a classical object of study in analysis. However, while their "global" behavior (i.e., the behavior in a neighborhood of infinity) was perfectly investigated long ago, many "local" questions (i.e., questions about their behavior near a finite point) have remained open until the present time. As an example we can name the problem of finding sharp estimates for the maximum of the modulus of an exponential polynomial on an interval I in terms of its maximum on a subset E of I having positive Lebesgue measure. This problem is solved in this article in the case where the measure of the set E is small compared with the length of I.

No less classical is another subject of this paper, the so-called uncertainty principle claiming that a function f and its Fourier transform $\widehat{f}$ cannot be small simultaneously (in an appropriate sense).

We shall show that one of the versions of this principle, namely, the Morgan theorem on the "strong annihilation" of two intervals on the real line (this terminology is borrowed from Jöricke and Havin's book "The uncertainty principle in harmonic analysis"), which is usually proved by analytic continuation technique, has quite "elementary" nature, and, moreover, elementary methods of proof work in a much more general case, where analytic continuation technique cannot be applied (namely, when the intervals are replaced by two sets of finite measure). A quantitative version of Zygmund's uniqueness theorem completes the article.

The following theorems may be regarded as main results of the paper.

Theorem I (the Turan lemma). *Let $p(t) = \sum_{k=1}^{n} c_k e^{\lambda_k t}$ $(c_k \in \mathbb{C},\ \lambda_k \in \mathbb{C})$ be an exponential polynomial of order n.* Let $I \subset \mathbb{R}$ be an interval, and E a measurable subset of I of positive measure.

Then

$$\sup_{t\in I} |p(t)| \leqslant e^{|I| \max|\operatorname{Re}\lambda_k|} \left\{ \frac{A\mu(I)}{\mu(E)} \right\}^{n-1} \sup_{t\in E} |p(t)|.$$

(Here and below A denotes an absolute constant).

Theorem II (the Amrein–Berthier theorem). *For every two sets E, $\Sigma \subset \mathbb{R}$ of finite* Lebesgue measure and every function $f \in L^2(\mathbb{R})$, the following inequality holds:

$$\|f\|^2_{L^2(\mathbb{R})} \leqslant Ae^{A\mu(E)\mu(\Sigma)} \left(\int_{\mathbb{R}\smallsetminus E} |f|^2 + \int_{\mathbb{R}\smallsetminus\Sigma} |\widehat{f}|^2 \right).$$

Theorem III (the Zygmund theorem). *Let a set $\Lambda = \{m_k\}_{k=-\infty}^{\infty} \subset \mathbb{Z}$ be lacunary in the sense of Zygmund*, i.e.,

$$\sup_{r\in\mathbb{Z}\smallsetminus\{0\}} \operatorname{card}\{\, (k', k'') : m_{k'} - m_{k''} = r \,\} \stackrel{\text{def}}{=} R(\Lambda) < +\infty.$$

Then for every function $f \in L^2(\mathbb{T})$ with spectrum $\operatorname{spec} f$ lying in Λ and for every subset E of the unit circumference $\mathbb{T}$ having positive measure, the following inequality holds:

$$\|f\|^2_{L^2(\mathbb{T})} \leqslant \exp\left\{ \frac{B(\varepsilon, R(\Lambda))}{\mu(E)^{2+\varepsilon}} \right\} \int_E |f|^2 \, d\mu.$$

Here $\varepsilon > 0$ can be taken arbitrary small, and $B(\varepsilon, R(\Lambda))$ depends only on ε and $R(\Lambda)$.

Corollary IV (The "logarithm summability" theorem). *Under the above assumptions on Λ, for every function $f \in L^2(\mathbb{T})$ such that $\operatorname{spec} f \subset \Lambda$ and f is not identically zero, and every $\varepsilon > 0$ we have*

$$\int_{\mathbb{T}} \log^{\frac{1}{2}-\varepsilon}\left(1 + \frac{1}{|f|}\right) d\mu < +\infty.$$

This selection is not indisputable and characterizes the paper only in the sense that the reader whose interest is not roused by the above results is unlikely to read the rest of the paper, and *vice versa*.

How is this paper organized?

Chapter 1, devoted to estimating exponential polynomials with purely imaginary exponents, is in the core of the article. Chapters 2 and 3, devoted to the Amrein–Berthier theorem and the Zygmund theorem, respectively, are almost independent. If the reader prefers to read Chapter 3 first, from Chapter 2 he will only need the lattice averaging lemma (Lemma 2.1). Chapter 4 contains commentaries, some supplements to the results of Chapters 1–3, and some open problems. I have tried to make the exposition as self-contained as possible. So, some readers may judge that the paper is overloaded with well-known classical results. I was fully aware of the latter, but, nevertheless, pursued the goal of making the topics in question attractive for people without (or with only a slight) knowledge of the subject. Moreover, I always thought that to skip a portion of a text might be much easier for the reader than to restore a skipped proof on the basis of the reference list.

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Notation list.

$\mathbb{Z}$ is the set of integers;

$\mathbb{N} = \{ z \in \mathbb{Z} : z > 0 \}$;

$\mathbb{R}$ is the real line;

$\mathbb{C}$ is the complex plane;

$\mathbb{C}_+ = \{ z \in \mathbb{C} : \operatorname{Im} z > 0 \}$ is the upper half-plane;

$\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ is the unit disk;

$\mathbb{T} = \partial\mathbb{D}$ is the unit circumference;

μ is the Lebesgue measure on $\mathbb{R}$ or on $\mathbb{T}$ normalized by $\mu([0,1]) = 1$ and $\mu(\mathbb{T}) = 1$, respectively. This normalization ensures that the standard mapping from $[0,1]$ onto $\mathbb{T}$ defined by $z = e^{2\pi i t}$ preserves the measure;

$|I|$ is the length of an interval $I \subset \mathbb{R}$ or an arc $I \subset \mathbb{T}$. In the first case $|I| = \mu(I)$, while in the second $|I| = 2\pi\mu(I)$.

The Lebesgue and Hardy spaces L^p and H^p, respectively, have their usual sense.

We denote by $\widehat{f}$ the Fourier transform of a function $f \in L^2(\mathbb{R})$ understood in the sense of the Plancherel theorem, i.e., as a limit in $L^2(\mathbb{R})$ of the functions

$$\widehat{f}_n(\lambda) \stackrel{\text{def}}{=} \int_{-n}^{n} f(x)e^{-2\pi i\lambda x}\,dx$$

I prefer to write 2π in the exponent, so, for the reader who is accustomed to the more widespread (though less convenient) definition containing the factor $1/\sqrt{2\pi}$ before the integral, some numerical constants will need correction. The inverse Fourier transform is denoted by $\check{f}$.

We shall not distinguish between 1-periodic functions defined on the real line and functions defined on $\mathbb{T}$. In either case we denote by $\widehat{f}_k$ the k-th Fourier coefficient of a function f:

$$\widehat{f}_k = \int_0^1 f(t)e^{-2\pi ikt}\,dt = \int_{\mathbb{T}} f(z)z^{-k}\,d\mu(z).$$

In Chapter 3, sometimes, we shall write $\widehat{f}(k)$ instead of $\widehat{f}_k$.

$\operatorname{supp} f = \{\, f \neq 0 \,\}$ is the support of a function f. If $f \in L^p(\mathbb{R})$, the support is defined up to a set of measure zero;

$\operatorname{spec} f = \operatorname{supp} \widehat{f}$ is the spectrum of a function f;

$\mathbf{P}\{A\}$ is the probability of a random event A;

$\mathbf{E}\xi$ is the mathematical expectation of a random variable ξ.

Notation connected with exponential polynomials. By an exponential polynomial we understand a finite sum

$$p(t) = \sum_{k=1}^{n} c_k e^{\lambda_k t} \qquad (c_k \in \mathbb{C},\ \lambda_k \in \mathbb{C}).$$

The number of nonvanishing terms in this sum is called the *order* of an exponential polynomial $p(t)$ and is denoted by $\operatorname{ord} p$. The numbers c_k are the coefficients of p, and the numbers λ_k are its exponents. The set of all exponents of an exponential polynomial p is called its spectrum and is denoted by $\operatorname{spec} p$. If all the exponents are purely imaginary, we shall sometimes change this notation and define the spectrum of p by $\operatorname{spec} p = \{\, \frac{\lambda_k}{2\pi i} : k = 1, \dots, n \,\}$; this set is the spectrum of the polynomial p treated as a distribution on the real line $\mathbb{R}$. The choice of definition will always be either inessential or clear from the context.

Some authors define an exponential polynomial as a more general expression $p(t) = \sum_{k=1}^{n} q_k(t)e^{\lambda_k t}$, where $q_k(t)$ are algebraic polynomials of degrees $\deg q_k = d_k$. It is natural to define the order of such an exponential polynomial as the sum $\operatorname{ord} p = \sum_{k=1}^{n}(d_k + 1)$. An obvious approximation argument shows that all the results of Chapter 1 remain valid in this more general setting.

C hapter 1. Estimates of exponential polynomials

Our main purpose in this chapter is to prove the inequality

$$\sup_{t\in I} |p(t)| \leqslant \left\{ \frac{A\mu(I)}{\mu(E)} \right\}^{n-1} \sup_{t\in E} |p(t)|, \tag{1.1}$$

where $p(t) = \sum_{k=1}^{n} c_k e^{i\lambda_k t}$ is an exponential polynomial with purely imaginary exponents, $I \subset \mathbb{R}$ is an interval, $E \subset I$ is a measurable set of positive Lebesgue measure, and A is an absolute constant.

§1.1. Historical remarks

In this section we are going to show what could have been well known by now if the fate would not have decided otherwise. For the first time estimate (1.1) in the special case where E is also an interval was derived by Turan from certain inequalities for sums of powers of n complex numbers (see [11]). Namely, Turan proved the following lemma.

Lemma. *Let $z_1, \dots, z_n$ be complex numbers*, $|z_j| \geqslant 1$, $j = 1, \dots, n$. Let

$$b_1, \dots, b_n \in \mathbb{C}, \qquad S_m \overset{\text{def}}{=} \sum_{k=1}^{n} b_k z_k^m \quad (m \in \mathbb{Z}).$$

Then

$$|S_0| \leqslant n\left\{\frac{2e(m+n-1)}{n}\right\}^{n-1} \max_{k=m+1}^{m+n} |S_k| \leqslant \left\{\frac{4e(m+n-1)}{n}\right\}^{n-1} \max_{k=m+1}^{m+n} |S_k| \tag{1.2}$$

for all $m \in \mathbb{Z}_+$.

Turan's proof was purely algebraic and was based on a construction of a polynomial $q(z) = 1 + \sum_{k=1}^{n} \gamma_k z^{m+k}$ such that $q(z_j) = 0$ for each $j = 1, \dots, n$ and $\sum_{k=1}^{n} |\gamma_k| \leqslant n\left\{\frac{2e(m+n-1)}{n}\right\}^{n-1}$. Obviously, in this case $S_0 = -\sum_{k=1}^{n} \gamma_k S_{m+k}$, whence (1.2) follows immediately. The polynomial $q(z)$ is exhibited explicitly, as a product $q(z) = \prod_{k=1}^{n}\bigl(1 - \frac{z}{z_k}\bigr)\sigma_m(z)$, where $\sigma_m(z)$ is the m-th partial sum of the series $\prod_{k=1}^{n}\bigl(1 - \frac{z}{z_k}\bigr)^{-1} = \sum_{k=0}^{\infty} \beta_k z^k$. The fact that the coefficients at the powers $z, z^2, \dots, z^m$ vanish follows from a simple observation that the s-th coefficient in the expansion of the product $1 \equiv \prod_{j=1}^{n}\bigl(1 - \frac{z}{z_j}\bigr)\sum_{k=0}^{\infty} \beta_k z^k$ depends only on $\beta_{s-n}, \dots, \beta_s$. The terms of the series $\sum_{k=0}^{\infty} \beta_k z^k$ are majorized by those of $(1-z)^{-n} = \sum_{k=0}^{\infty} \frac{(k+n-1)!}{k!\,(n-1)!} z^k$. Thus, all coefficients of $\sigma_m(z)$ do not exceed $\frac{(m+n-1)!}{m!\,(n-1)!} \leqslant \left\{\frac{e(m+n-1)}{n}\right\}^{n-1}$. The coefficients of the polynomial $\prod_{j=1}^{n}\bigl(1 - \frac{z}{z_j}\bigr)$ are not greater than $\binom{n}{k}$. Carefully examining everything contributing to γ_k, we get the estimates

$$|\gamma_k| \leqslant \left\{\frac{e(m+n-1)}{n}\right\}^{n-1} \sum_{s=k}^{n} \binom{n}{s}$$

and

$$\sum_{k=1}^{n} |\gamma_k| = \frac{1}{2}\sum_{k=1}^{n}\bigl(|\gamma_k| + |\gamma_{n+1-k}|\bigr) \leqslant 2^{n-1} n\left\{\frac{e(m+n-1)}{n}\right\}^{n-1},$$

which prove (1.2).

It follows immediately from (1.2) that

$$|p(t_0)| \leqslant \left\{\frac{4e(m+n-1)}{n}\right\}^{n-1} \max_{k=m+1}^{m+n} |p(t_k)|$$

if the points $t_k = t_0 + k\delta$ form an arithmetic progression (it suffices to put $z_j = e^{i\lambda_j\delta}$, $b_j = c_j e^{i\lambda_j t_0}$); now inequality (1.1) for the case where E is an interval can be derived in an almost immediate way (with the constant $A = 4e$).

Turan himself said no word about the case of an arbitrary measurable set E. But using the same idea in its simplest form (with $m = 0$, which corresponds to the polynomial $q(z) = \prod_{k=1}^{n}\big(1 - \frac{z}{z_k}\big)$ and to the estimate $|S_0| \leqslant 2^n \max_{k=1}^{n} |S_k|$), one can prove that

$$\max_{t\in I} |p(t)| \leqslant 2^n \left\{ \frac{\mu(I)}{\mu(E)} \right\}^{2n^2} \max_{t\in E} |p(t)|. \tag{1.3}$$

Indeed, the inequality $|p(t_0)| \leqslant 2^n \max_{t\in E} |p(t)|$ holds if t_0 is the first term of an arithmetic progression $t_k = t_0 + k\delta$ $(k = 0, 1, \ldots, n)$ with all other terms belonging to E. It is quite natural to think that such a progression exists for the points $t_0 \in I$ near which E is "sufficiently dense". We shall see that all points t_0 for which there is an open interval J satisfying $t_0 \in J \subset I$ and

$$\mu(E \cap J) > \left(1 - \frac{1}{2n}\right)\mu(J) \tag{$*$}$$

will fit.

Indeed, such a point t_0 splits the interval J into two subintervals J_- and J_+. At least for one of them (*) is also valid. Assume, for certainty, that $\mu(E \cap J_+) > \big(1 - \frac{1}{2n}\big)\mu(J_+)$. Applying the lattice averaging lemma (Lemma 2.1 of Chapter 2), we see that the average number of points $t_k = t_0 + k\delta$ $(k \in \mathbb{N})$ belonging to $J_+ \setminus E$ as δ runs over the interval $\big(\frac{\mu(J_+)}{2n}, \frac{\mu(J_+)}{n}\big)$ is less than 1. Hence, there exists a positive value $\delta < \frac{\mu(J_+)}{n}$ such that none of the points $t_1, \ldots, t_n$ belongs to $J_+ \setminus E$. But all these points lie in J_+ and, consequently, in E.

If the reader deems the lattice averaging lemma little-known or nontrivial, he may restrict himself to the density estimate $\mu(E \cap J) > \big(1 - \frac{1}{n^2}\big)\mu(J)$, which simplifies the proof: one needs only to estimate the measure of the set of all $\delta \in \big[0, \frac{\mu(J_+)}{n}\big]$ such that $t_k \in J_+ \setminus E$ for each k separately. In this case the exponent $2n^2$ should be replaced by n^3.

Now let $E_1 = \bigcup\big\{\, J : J \subset I \text{ is open, } \mu(E \cap J) > \big(1 - \frac{1}{2n}\big)\mu(J) \,\big\}$. Clearly, the open set E_1 contains all density points of E, so $E \subset E_1$ up to a set of measure zero. If $E_1 \neq I$, then for each constituent interval Q of E_1 the inequality $\mu(E \cap Q) \leqslant \big(1 - \frac{1}{2n}\big)\mu(Q)$ holds (otherwise E_1 would include some interval strictly larger than Q). Thus, the set E_1 has the following two properties:

$$\sup_{t\in E_1} |p(t)| \leqslant 2^n \sup_{t\in E} |p(t)|, \tag{A$_1$}$$

$$\mu(E_1) \geqslant \big(1 - \frac{1}{2n}\big)^{-1}\mu(E) \geqslant e^{1/(2n)}\mu(E) \quad \text{or} \quad E_1 = I. \tag{B$_1$}$$

Iterating this procedure (constructing E_2 starting with E_1 in the same way and so on), we obtain a sequence of sets $E_1, E_2, \ldots$ such that

$$\sup_{t\in E_1} |p(t)| \leqslant 2^{nk} \sup_{t\in E} |p(t)|, \tag{A$_k$}$$

$$\mu(E_k) \geqslant e^{k/(2n)}\mu(e) \quad \text{or} \quad E_k = I. \tag{B$_k$}$$

If $k > 2n\log\frac{\mu(I)}{\mu(E)}$, then the first case in (B_k) cannot occur; therefore, $E_{\left[2n\log\frac{\mu(I)}{\mu(E)}+1\right]} = I$, whence

$$\sup_{t\in I}|p(t)| \leqslant 2^{\left(2n\log\frac{\mu(I)}{\mu(E)}+1\right)n}\sup_{t\in E}|p(t)| \leqslant 2^n\left\{\frac{\mu(I)}{\mu(E)}\right\}^{2n^2}\sup_{t\in E}|p(t)|.$$

As far as the leading idea of this reasoning (estimating along arithmetic progressions) was suggested by Turan and all the remaining part of the proof follows from this main idea in quite a natural way and is nothing but mathematical routine, one should regard inequality (1.3) to be due to Turan and date it back to 1953.

Often it is desirable to have an upper estimate of the sum of absolute values of coefficients $\sum_{k=1}^n|c_k|$ rather than of the maximum $\max_{t\in I}|p(t)|$. In this case the "separation condition" $\lambda_{j+1}-\lambda_j \geqslant \Delta > 0$ ($j = 1,\dots,n-1$) is usually imposed on the exponents $\lambda_1 < \lambda_2 < \dots < \lambda_n \in \mathbb{R}$. A desired estimate can be derived from (1.1) (or (1.3)) by using the well-known Salem inequality: if $\mu(I) \geqslant 4\pi/\Delta$, then

$$\frac{1}{n}\left(\sum_{k=1}^n|c_k|\right)^2 \leqslant \sum_{k=1}^n|c_k|^2 \leqslant \frac{4}{|I|}\int_I|p(t)|^2\,dt \leqslant 4\max_{t\in I}|p(t)|^2.$$

If the interval I is too small to apply the Salem inequality, we can take a wider interval $I' \supset I$ of length $4\pi/\Delta$. So, we get

$$\sum_{k=1}^n|c_k| \leqslant \begin{cases}\left\{\dfrac{A\mu(I)}{\mu(E)}\right\}^{n-1}\max_{t\in E}|p(t)| & \text{if } \Delta\mu(I)\geqslant 4\pi,\\[2ex] \left\{\dfrac{4\pi A}{\Delta\mu(E)}\right\}^{n-1}\max_{t\in E}|p(t)| & \text{if } \Delta\mu(I)< 4\pi,\end{cases} \tag{1.1$'$}$$

where A is an absolute constant (slightly greater than that in (1.1)).

Concerning the state of affairs in the reality, I do not know any publication containing an estimate essentially better than

$$\sup_{t\in I}|p(t)| \leqslant \left\{\frac{2\mu(I)}{\mu(E)}\right\}^{2^n}\sup_{t\in E}|p(t)|.$$

§1.2. Two useful lemmas

This section contains the proofs of two propositions frequently used in the rest of the paper, namely, a weak type estimate of the logarithmic derivative of a polynomial on the unit circumference and on the real line, and the Langer lemma on the distribution of zeros of exponential polynomials. The first of them is a version of the Kolmogorov inequality for the harmonic conjugate function and is aimed merely at calculating certain constants. The second one has been included because it is relatively little-known. Here are the precise statements.

Lemma 1.2. *If $P(z)$ is an algebraic polynomial of degree n, then*

$$\mu\Big\{\, x \in \mathbb{R} : \Big|\frac{d}{dx}\log P(x)\Big| > y \,\Big\} \leqslant \frac{8n}{y}$$

and

$$\mu\Big\{\, z \in \mathbb{T} : \Big|\frac{d}{dz}\log P(z)\Big| > y \,\Big\} \leqslant \frac{8n}{\pi y}.$$

Remark. It is worth noting that $\frac{d}{dz}\log P(z) = \frac{\frac{d}{dz}P(z)}{P(z)}$, and, therefore, one can regard the inequalities of Lemma 1.2 as certain versions of the Bernstein estimate for the derivative of an algebraic polynomial $P(z)$.

Lemma 1.3. *Let $p(z) = \sum_{k=1}^{n} c_k e^{i\lambda_k z}$ $(0 = \lambda_1 < \lambda_2 < \cdots < \lambda_n = \lambda)$ be an exponential polynomial not vanishing identically. Then the number of complex zeros of $p(z)$ in an open vertical strip $x_0 < \operatorname{Re} z < x_0 + \Delta$ of width Δ does not exceed $(n-1) + \frac{\lambda\Delta}{2\pi}$.*

Proof of Lemma 1.2. First we shall prove the inequality for the real line. Let $z_1, \dots, z_{n_1}$; $\zeta_1, \dots, \zeta_{n_2}$ $(n_1 + n_2 = n)$ be complex zeros of the polynomial P enumerated in such a way that $\operatorname{Im} z_j \leqslant 0$ $(j = 1, \dots, n_1)$, $\operatorname{Im} \zeta_j > 0$ $(j = 1, \dots, n_2)$. We have

$$\frac{d}{dz}\log P(z) = \sum_{j=1}^{n_1} \frac{1}{z - z_j} + \sum_{j=1}^{n_2} \frac{1}{z - \zeta_j} = \textstyle\sum_1(z) + \sum_2(z).$$

The function $\sum_1(z)$ is analytic in the upper half-plane $\mathbb{C}_+$, and

$$\operatorname{Im} \textstyle\sum_1(z) = \displaystyle\sum_{j=1}^{n_1} \frac{\operatorname{Im} z_j - \operatorname{Im} z}{|z - z_j|^2} < 0$$

for all $z \in \mathbb{C}_+$.

Let $h(\xi)$ be the harmonic measure of the set $\mathbb{R} \setminus [-y, y]$ with respect to the upper half-plane and a point $\xi \in \mathbb{C}_+$. Repeating an argument of Koosis, we put $u(z) \overset{\text{def}}{=} h(-\sum_1(z))$. The function $u(z)$ is harmonic in $\mathbb{C}_+$, $0 \leqslant u(z) \leqslant 1$, $u(it) \xrightarrow[t \to +\infty]{} 0$, and $u(z) \geqslant 1/2$ if $|\sum_1(z)| \geqslant y$ (the latter fact follows from the geometric description of the harmonic measure as a ratio to π of the angle at which a subset of $\mathbb{R}$ is seen from the point ξ).

Moreover, we have

$$\lim_{t \to +\infty} \pi t u(it) = \int_{\mathbb{R}} u(x)\, dx \geqslant \frac{1}{2}\mu\big\{\, x \in \mathbb{R} : |\textstyle\sum_1(x)| > y \,\big\}.$$

On the other hand, an easy computation shows that

$$\lim_{t \to +\infty} \pi t u(it) = \lim_{t \to +\infty} \pi t h\Big(\frac{i n_1}{t} + O\big(\frac{1}{t^2}\big)\Big) = \frac{2n_1}{y}.$$

Hence

$$\mu\{\, x\in\mathbb{R} : |\textstyle\sum_1(x)| > y \,\} \leqslant \frac{4n_1}{y}.$$

Similarly,

$$\mu\{\, x\in\mathbb{R} : |\textstyle\sum_2(x)| > y \,\} \leqslant \frac{4n_2}{y}.$$

Combining these estimates, we obtain

$$\mu\{\, x\in\mathbb{R} : |\textstyle\sum(x)| > y \,\} \leqslant \mu\{\, x\in\mathbb{R} : |\sum_1| > \dfrac{n_1}{n}y \,\} + \mu\{\, x\in\mathbb{R} : |\sum_2| > \dfrac{n_2}{n}y \,\} \leqslant \dfrac{8n}{y},$$

as desired.

Now we pass to the case of the circumference. As above, we split the zeros of $P(z)$ into two groups: $z_1,\dots,z_{n_1}\in\mathbb{D}$ and $\zeta_1,\dots,\zeta_{n_2}\in\mathbb{C}\setminus\mathbb{D}$. Then

$$\frac{d}{dz}\log P(z) = \frac{1}{z}\left(\sum_{j=1}^{n_1}\frac{z}{z-z_j} + \sum_{j=1}^{n_2}\frac{z}{z-\zeta_j}\right) = \frac{1}{z}\big(\textstyle\sum_1(z)+\sum_2(z)\big).$$

The factor $1/z$ can be disregarded since its absolute value is equal to 1. The estimate for $\sum_1(z)$ is essentially the same as above: having established the inequality

$$\operatorname{Re}\textstyle\sum_1(z) = \displaystyle\sum_{j=1}^{n_1}\frac{|z|^2-\operatorname{Re} z\overline{z}_j}{|z-z_j|^2} \geqslant 0$$

for $z\in\mathbb{C}\setminus\mathbb{D}$, we consider the function $u(z)\overset{\text{def}}{=}h\big(i\sum_1(z)\big)$, which is harmonic outside the unit disk, and derive the estimate

$$u(\infty) = \frac{2\operatorname{arctg}\frac{n_1}{y}}{\pi} = \int_{\mathbb{T}} u(z)\,d\mu(z) \geqslant \frac{1}{2}\mu\{\, z\in\mathbb{T} : |\textstyle\sum_1(z)| > y \,\},$$

which implies

$$\mu\{|\textstyle\sum_1(z)| > y\} \leqslant \dfrac{4}{\pi}\operatorname{arctg}\dfrac{n_1}{y} \leqslant \dfrac{4}{\pi}\dfrac{n_1}{y}.$$

The function $\operatorname{Re}\sum_2(z)$ may change sign. Therefore, we use another inequality

$$\operatorname{Re}\sum_2(z) = n_2 - \sum_{j=1}^{n_2}\frac{|\zeta_j|^2-\operatorname{Re} z\overline{\zeta}_j}{|z-\zeta_j|^2} \leqslant n_2 \qquad (z\in\mathbb{D}).$$

This time we choose the function h (see above) to be harmonic in $\mathbb{C}_+ - in_2$. In order to obtain the estimate $\mu\{|\sum_2| > y\} \leqslant \frac{4}{\pi}\frac{n_1}{y}$, we can restrict ourselves to values $y > n_2$. Let $h(\xi)$ be the harmonic measure of $(\mathbb{R}-in_2)\setminus I$ (where I is the interval cut off from the line $\mathbb{R}-in_2$ by the circle centered at 0 and of radius y) with respect to the half-plane $\mathbb{C}_+ - in_2$ and the point $\xi\in\mathbb{C}_+ - in_2$. Drawing a picture, one can easily check that the function $u(z) \stackrel{\text{def}}{=} h(-i\sum_2(z))$ is harmonic in $\mathbb{D}$, $u(z) \geqslant \frac{\pi - \arccos\frac{n_2}{y}}{\pi}$ if $|\sum_2(z)| > y$, and $u(0) = h(0) = \frac{2\arcsin\frac{n_2}{y}}{\pi}$; therefore,

$$\mu\{\, |\textstyle\sum_2| > y \,\} \leqslant \frac{2\arcsin\frac{n_2}{y}}{\pi - \arccos\frac{n_2}{y}} = \frac{2\arcsin\frac{n_2}{y}}{\pi + \arcsin\frac{n_2}{y}}.$$

Now, to get the desired estimate it suffices to verify that $\frac{2\theta}{\pi/2+\theta} \leqslant \frac{4}{\pi}\sin\theta$ for each $\theta \in [0, \frac{\pi}{2}]$. The last inequality is equivalent to $\frac{\sin\theta}{\theta} + \frac{2}{\pi}\sin\theta \geqslant 1$. Taking into account that $\sin\theta \geqslant \theta - \frac{1}{6}\theta^3$ for every $\theta \geqslant 0$ and $\sin\theta \geqslant \frac{2}{\pi}\theta$ for every $\theta \in [0, \frac{\pi}{2}]$, we have

$$\frac{\sin\theta}{\theta} + \frac{2}{\pi}\sin\theta \geqslant 1 - \frac{1}{6}\theta^2 + \frac{4}{\pi^2}\theta = 1 + \theta\Big(\frac{4}{\pi^2} - \frac{1}{6}\theta\Big) \geqslant 1 + \theta\Big(\frac{4}{\pi^2} - \frac{\pi}{12}\Big),$$

and it remains to notice that $\pi^3 \leqslant 48$.

As above, the estimate of $\sum(z)$ results from the estimates of $\sum_1(z)$ and $\sum_2(z)$. Lemma 1.2 is proved.

Proof of lemma 1.3. Without loss of generality we assume that the coefficients c_1 and c_2 do not vanish and the boundary of the strip $x_0 < \operatorname{Re} z < x_0 + \Delta$ is free of zeros of the exponential polynomial $p(z)$. Following the reasoning of Langer, we make use of the argument principle to estimate the number of zeros of $p(z)$ in the rectangle $Q = \{\, z : x_0 < \operatorname{Re} z < x_0 + \Delta,\ |\operatorname{Im} z| \leqslant y \,\}$, as $y \to +\infty$.

On the upper edge of Q we have $p(z) = c_1 + O(e^{-\lambda_2 y})$. Therefore, the argument increment along this edge tends to 0 as $y \to +\infty$. Similarly, the representation $p(z) = c_n e^{i\lambda z}(1 + O(e^{-(\lambda-\lambda_{n-1})y}))$, which is valid on the lower edge of Q, implies that the argument increment along the lower edge tends to $\lambda\Delta$ as $y \to +\infty$.

We show that the argument increment along any vertical segment $\{\, z = x + it : t \in [\alpha, \beta] \,\}$ free of zeros of $p(z)$ does not exceed $\pi(n-1)$. Let $\xi \stackrel{\text{def}}{=} e^{i \arg p(x_0 + i\alpha)}$. The function $q(t) \stackrel{\text{def}}{=} \operatorname{Im} \overline{\xi} p(x_0 + it) = \sum_{k=1}^n a_k e^{-\lambda_k t}$ ($a_k = \operatorname{Im} \overline{\xi} c_k e^{i\lambda_k x_0} \in \mathbb{R}$) is a real exponential polynomial. The definition of ξ ensures $q(\alpha) = 0$. If $q \equiv 0$, then all values $p(x_0 + it)$ ($t \in [\alpha, \beta]$) lie on the ray $\{\, \xi y : y > 0 \,\}$, and therefore $\Delta_{[\alpha,\beta]} \arg p(x_0 + it) = 0$. Otherwise, real zeros of $q(t)$ split the segment $[\alpha, \beta]$ into at most $n-1$ intervals I_j (it is well known that a real exponential polynomial of order n has at most $n-1$ real zeros). Within each of the intervals I_j, the values $p(x_0 + it)$ lie in one of the two half-planes generated by the line $\{\, \xi y : y \in \mathbb{R} \,\}$, whence $|\Delta_{I_j} \arg p(x_0 + it)| \leqslant \pi$. Adding these inequalities, we obtain $|\Delta_{[\alpha,\beta]} \arg p(x_0 + it)| \leqslant \pi(n-1)$, which proves the desired statement. In particular, the argument increment along each of the lateral edges of Q does not exceed $\pi(n-1)$. So the total argument increment of $p(z)$ along the boundary of Q traced counter-clockwise can be estimated from above by a quantity tending to $2\pi\big(\frac{\Delta\lambda}{2\pi} + (n-1)\big)$ as $y \to +\infty$, whence Lemma 1.3 clearly follows.

§1.3. The proof of estimate (1.1) in a special case

Here we shall prove inequality (1.1) for the case of a 1-periodic exponential polynomial $p(t) = \sum_{k=1}^n c_k e^{2\pi i m_k t}$, where $c_k \in \mathbb{C}$, $m_1 < \cdots < m_n \in \mathbb{Z}$, and for the segment $I =$

$[0,1]$. This case is singled out for two reasons: firstly, here the main idea of the proof is not obscured by inessential technical details and may be seen clearer, and, secondly, the constant A in (1.1) can be taken in this case much smaller than in the general case. Substituting $z = e^{2\pi it}$, we come to the following statement.

Theorem 1.4 (the Turan lemma for polynomials on the unit circumference). *Let* $p(z) = \sum_{k=1}^{n} c_k z^{m_k}$ ($c_k \in \mathbb{C}$, $m_1 < \dots < m_n \in \mathbb{Z}$) *be a trigonometric polynomial on the unit circumference* $\mathbb{T}$, *and let* E *be a measurable subset of* $\mathbb{T}$. *Then*

$$\|p\|_W \stackrel{\text{def}}{=} \sum_{k=1}^{n} |c_k| \leqslant \left\{\frac{16e}{\pi}\frac{1}{\mu(E)}\right\}^{n-1} \sup_{z\in E}|p(z)| \leqslant \left\{\frac{14}{\mu(E)}\right\}^{n-1} \sup_{z\in E}|p(z)|.$$

(Here W stands in honor of Wiener, as usual).

Proof. We shall construct by induction a sequence of polynomials $p_n, p_{n-1}, \dots, p_1$ such that

1) $p_n = p$;
2) $\operatorname{ord} p_k = k$ ($k = 1, \dots, n$);
3) $\|p_{k-1}\|_W \geqslant \frac{\pi}{16}\|p_k\|_W$;
4) the ratio $\varphi_k \stackrel{\text{def}}{=} \left|\frac{p_{k-1}}{p_k}\right|$ admits the weak type estimate $\mu\{\, z \in \mathbb{T} : \varphi_k(z) > t \,\} \leqslant 1/t$ for all $t > 0$.

The construction is as follows. Let $p_n = p$. The polynomial $p_k(z) = \sum_{s=1}^{k} d_s z^{r_s}$ ($r_1 < \dots < r_k \in \mathbb{Z}$) being chosen, we introduce two polynomials

$$\underline{q}(z) \stackrel{\text{def}}{=} \frac{d}{dz}(z^{-r_1} p_k(z)) \quad \text{and} \quad \overline{q}(z) \stackrel{\text{def}}{=} \frac{d}{dz}(z^{-r_k} p_k(z)).$$

Obviously, $\operatorname{ord}\underline{q} = \operatorname{ord}\overline{q} = k-1$. We have

$$\|\underline{q}\|_W = \sum_{s=1}^{k} |d_s|(r_s - r_1), \qquad \|\overline{q}\|_W = \sum_{s=1}^{k} |d_s|(r_k - r_s),$$

whence

$$\|\underline{q}\|_W + \|\overline{q}\|_W = (r_k - r_1)\sum_{s=1}^{k} |d_s| = r\|p_k\|_W,$$

where $r \stackrel{\text{def}}{=} r_k - r_1$. So, at least one of the norms $\|\underline{q}\|_W$ and $\|\overline{q}\|_W$ is at least $\frac{r}{2}\|p_k\|_W$. We assume, for certainty, that $\|\overline{q}\|_W \geqslant \frac{r}{2}\|p_k\|_W$ (the other case is similar). Put $p_{k-1}(z) = \frac{\pi}{8r}\overline{q}(z)$. Conditions 2) and 3) are satisfied, and we must check condition 4) only. We observe that $z^{-r_k} p_k(z) = g(1/z)$, where g is an algebraic polynomial of degree r. But then $\overline{q}(z) = -\frac{1}{z^2} g'\left(\frac{1}{z}\right)$, and

$$\mu\{\, z \in \mathbb{T} : \varphi_k(z) > t \,\} = \mu\left\{\, z \in \mathbb{T} : \left|\frac{g'(1/z)}{g(1/z)}\right| > \frac{8r}{\pi} t \right\} \leqslant \frac{1}{t}$$

because of Lemma 1.2 and the simple observation that the substitution $z \to 1/z$ preserves Lebesgue measure on the unit circumference. Now we estimate the measure of the set of all points $z \in \mathbb{T}$ for which $|p_1(z)|$ is essentially greater than $|p_n(z)| = |p(z)|$. We have

$$\left|\frac{p_1(z)}{p_n(z)}\right| = \prod_{k=2}^{n} \varphi_k(z) \leqslant \exp\left\{\sum_{k=2}^{n} \psi_k(z)\right\},$$

where $\psi_k(z) \stackrel{\text{def}}{=} \log_+ \varphi_k(z)$. The weak type estimate of φ_k gives the inequality $\mu\{\psi_k > t\} \leqslant e^{-t}$ for all $t > 0$. Let $\alpha > 0$. We decompose $\psi_k(z)$ into the sum of $\eta_k(z) \stackrel{\text{def}}{=} \min(\psi_k(z), \alpha)$ and $\varkappa_k(z) \stackrel{\text{def}}{=} \psi_k(z) - \eta_k(z)$. Then $\sum_{k=2}^{n} \eta_k(z) \leqslant \alpha(n-1)$ for all $z \in \mathbb{T}$, and

$$\int_{\mathbb{T}} \varkappa_k(z)\, d\mu(z) = \int_{\alpha}^{+\infty} \mu\{\psi_k > t\}\, dt \leqslant \int_{\alpha}^{+\infty} e^{-t}\, dt = e^{-\alpha}.$$

Therefore,

$$\int_{\mathbb{T}} \left(\sum_{k=2}^{n} \varkappa_k(z)\right) d\mu(z) \leqslant e^{-\alpha}(n-1)$$

and

$$\mu\left\{ z \in \mathbb{T} : \sum_{k=2}^{n} \psi_k(z) > (\alpha+1)(n-1) \right\} \leqslant \mu\left\{ z \in \mathbb{T} : \sum_{k=2}^{n} \varkappa_k(z) > n-1 \right\} < e^{-\alpha}.$$

Let $\alpha = \log \frac{1}{\mu(E)}$; then $e^{-\alpha} = \mu(E)$ and there exists a point $z_0 \in E$ for which $\sum_{k=2}^{n} \psi_k(z_0) \leqslant (\alpha+1)(n-1)$. Now we have

$$\begin{aligned}\left(\frac{\pi}{16}\right)^{n-1} \|p\|_W &\leqslant \|p_1\|_W \stackrel{(\operatorname{ord} p_1 = 1!)}{=} |p_1(z_0)| \\ &\leqslant \exp\left\{\left(1 + \log\frac{1}{\mu(E)}\right)(n-1)\right\} |p(z_0)| = \left\{\frac{e}{\mu(E)}\right\}^{n-1} |p(z_0)| \\ &\leqslant \left\{\frac{e}{\mu(E)}\right\}^{n-1} \sup_{z\in E} |p(z)|,\end{aligned}$$

and the theorem is proved.

§1.4. The proof of the Turan lemma in the general case

Theorem 1.5 (the Turan lemma). *Let* $p(t) = \sum_{k=1}^{n} c_k e^{i\lambda_k t}$, *where* $c_k \in \mathbb{C}$ *and* $\lambda_1 < \cdots < \lambda_n \in \mathbb{R}$. *If* E *is a measurable subset of the segment* $I = [-\frac{1}{2}, \frac{1}{2}]$, *then*

$$\sup_{t\in I} |p(t)| \leqslant \left\{\frac{316}{\mu(E)}\right\}^{n-1} \sup_{t\in E} |p(t)|.$$

(It is clear that the restriction $I = [-\frac{1}{2}, \frac{1}{2}]$ is not essential. The case of an arbitrary interval I can easily be reduced to this special case by a suitable linear substitution).

Proof of Theorem 1.5. The development of the proof depends on the interrelation of $\lambda \stackrel{\text{def}}{=} \lambda_n - \lambda_1$ and $n-1$.

Case 1: $\lambda \leqslant n-1$. We shall see that in this case the exponential polynomial $p(t)$ behaves on the interval $[-\frac{1}{2}, \frac{1}{2}]$ like an algebraic polynomial of degree $n-1$ and satisfies the estimate

$$\sup_{t \in [-\frac{1}{2}, \frac{1}{2}]} |p(t)| \leqslant \left\{ \frac{154}{\mu(E)} \right\}^{n-1} \sup_{t \in E} |p(t)|.$$

If $n = 1$, the statement is obvious. Let $n > 1$. Without loss of generality, we assume that $0 = \lambda_1 < \cdots < \lambda_n = \lambda \leqslant n-1$. By virtue of the Langer lemma, complex zeros of the exponential polynomial $p(z)$ are well separated, namely, each vertical strip of width Δ contains at most $\frac{\Delta\lambda}{2\pi} + (n-1) \leqslant \left(1 + \frac{\Delta}{2\pi}\right)(n-1)$ zeros. Let us enumerate z_j in the order of increase of $|\operatorname{Re} z_j|$. For every $j \in \mathbb{N}$, the inequality $|\operatorname{Re} z_j| \geqslant \pi \frac{j-(n-1)}{(n-1)}$ holds (otherwise the zeros $z_1, \ldots, z_j$ would lie in a vertical strip of width $\Delta < 2\pi \frac{j-(n-1)}{(n-1)}$, but this strip can contain at most $\left(1 + \frac{\Delta}{2\pi}\right)(n-1) < j$ zeros). Now we write the Hadamard factorization of $p(z)$:

$$p(z) = ce^{az} \prod_{j=1}^{2(n-1)} (z - z_j) \prod_{j > 2(n-1)} (1 - \frac{z}{z_j}) e^{z/z_j} \stackrel{\text{def}}{=} ce^{az} Q(z) R(z).$$

We shall examine the behavior of each of the above three factors separately.

The canonical product $R(z)$.

First of all, we notice that $|\operatorname{Re} z_j| \geqslant \pi$ if $j > 2(n-1)$. We have (since $|\operatorname{Re} z| < \pi$):

$$\begin{aligned} \left| \frac{d}{dz} \log R(z) \right| &= \left| \sum_{j > 2(n-1)} \left(\frac{1}{z_j} + \frac{1}{z - z_j} \right) \right| \leqslant |z| \sum_{j > 2(n-1)} \frac{1}{|z_j||z - z_j|} \\ &\leqslant |z| \sum_{j > 2(n-1)} \frac{1}{|\operatorname{Re} z_j|(|\operatorname{Re} z_j| - |\operatorname{Re} z|)}. \end{aligned}$$

whence it follows, since $z \in [-\frac{1}{2}, \frac{1}{2}]$, that

$$\begin{aligned} \left| \frac{d}{dz} \log R(z) \right| &\leqslant |z| \sum_{j > 2(n-1)} \frac{1}{|\operatorname{Re} z_j|(|\operatorname{Re} z_j| - 1/2)} \leqslant 2|z| \sum_{j=1}^{\infty} \frac{1}{|\operatorname{Re} z_{2(n-1)+j}|^2} \\ &\leqslant 2|z| \sum_{j=1}^{\infty} \frac{1}{(\pi + \frac{\pi j}{n-1})^2} \leqslant \frac{2(n-1)}{\pi} |z| \int_{\pi}^{\infty} \frac{dt}{t^2} \\ &= \frac{2|z|}{\pi^2}(n-1). \end{aligned}$$

If z is purely imaginary, then

$$\left| \frac{d}{dz} \log R(z) \right| \leqslant |z| \sum_{j > 2(n-1)} \frac{1}{|\operatorname{Re} z_j|^2} \leqslant \frac{|z|}{\pi^2}(n-1).$$

Now,

$$\int_{-1/2}^{1/2}\left|\frac{d}{dz}\log R(z)\right|dz \leqslant \frac{2(n-1)}{\pi^2}\int_{-1/2}^{1/2}|z|\,dz = \frac{n-1}{2\pi^2},$$

and, therefore,

$$\max_{z\in[-\frac{1}{2},\frac{1}{2}]}|R(z)| \leqslant \exp\left\{\frac{n-1}{2\pi^2}\right\}\min_{z\in[-\frac{1}{2},\frac{1}{2}]}|R(z)|.$$

The factor ce^{az}.

The simplest way to estimate $|\operatorname{Re} a|$ is to consider the argument increment of $p(z)$ along a segment $[-i\omega, i\omega]$ ($\omega > 0$). It follows from the proof of the Langer lemma that $|\Delta_{[-i\omega,i\omega]}\arg p| \leqslant \pi(n-1)$. The argument increment brought in by each of the zeros of $Q(z)$ does not exceed π. So, we have

$$\begin{aligned}
|\Delta_{[-i\omega,i\omega]}\arg Q| &\leqslant 2\pi(n-1);\\
|\Delta_{[-i\omega,i\omega]}\arg R| &\leqslant \int_{-\omega}^{\omega}\left|\frac{d}{dz}\log R(it)\right|dt \leqslant \frac{n-1}{\pi^2}\int_{-\omega}^{\omega}|t|\,dt = \frac{n-1}{\pi^2}\omega^2;\\
\Delta_{[-i\omega,i\omega]}\arg(ce^{az}) &= 2\omega\operatorname{Re} a.
\end{aligned}$$

The identity

$$\Delta_{[-i\omega,i\omega]}\arg p = 2\omega\operatorname{Re} a + \Delta_{[-i\omega,i\omega]}\arg Q + \Delta_{[-i\omega,i\omega]}\arg R$$

implies

$$|\operatorname{Re} a| \leqslant \min_{\omega>0}\left(\frac{3\pi}{2\omega} + \frac{\omega}{2\pi^2}\right)(n-1) = \sqrt{\frac{3}{\pi}}(n-1).$$

Thus, the absolute values of all the factors except $Q(z)$ change only slightly on the interval $[-\frac{1}{2},\frac{1}{2}]$. More precisely,

$$\begin{aligned}
\max_{z\in[-\frac{1}{2},\frac{1}{2}]}\left|\frac{p(z)}{Q(z)}\right| &\leqslant \exp\left\{\left(\sqrt{\frac{3}{\pi}} + \frac{1}{2\pi^2}\right)(n-1)\right\}\min_{z\in[-\frac{1}{2},\frac{1}{2}]}\left|\frac{p(z)}{Q(z)}\right|\\
&\leqslant 3^{n-1}\min_{z\in[-\frac{1}{2},\frac{1}{2}]}\left|\frac{p(z)}{Q(z)}\right|.
\end{aligned}$$

It remains to examine

The behavior of the polynomial $Q(z)$.

Let $0 < h < 1/8$. We shall carry out the Cartan lemma construction. Let n_1 be the maximal integer for which there exists a disk D_1 of radius $\frac{n_1}{n-1}h$ containing at least n_1 zeros of the polynomial Q (it is clear that D_1 contains exactly n_1 zeros of Q because otherwise n_1 could be enlarged). Let n_2 be the maximal integer for which there exists a disk D_2 of radius $\frac{n_2}{n-1}h$ containing at least n_2 zeros of Q among those not lying in D_1, and so on, till all the zeros of Q are covered. Putting $D'_k = 2D_k$ (i.e., the disk centered at the same point and of double radius), we obtain the sequence of integers $n_1 \geqslant \cdots \geqslant n_s$ with the sum $n_1 + \cdots + n_s = 2(n-1)$ and the corresponding sequence of disks $D'_1, \dots, D'_s$ with the sum of radii equal to $4h$. We fix a point $z \in [-\frac{1}{2}, \frac{1}{2}] \setminus \bigcup_{k=1}^{s} D'_k$ and enumerate the zeros of Q in the order of increase of $|z - z_j|$. Following Cartan, we shall show that $|z - z_j| \geqslant \frac{j}{n-1} h$. Indeed, if this is not the case, then the disk D centered at z and of radius $\frac{j}{n-1} h$ contains at least j zeros of Q. Choose an $m \in \{1, \dots, s\}$ such that $n_1 \geqslant \dots \geqslant n_m \geqslant j > n_{m+1} \geqslant \dots \geqslant n_s$. For every $z \notin \bigcup_{k=1}^{s} D'_k$ the distance between z and the center of D_k is at least $\frac{2n_k}{n-1} h \geqslant \frac{n_k}{n-1} h + \frac{j}{n-1} h$ if $k \leqslant m$. Hence, D does not intersect any of the disks $D_1, \dots, D_m$. But if this were true, the disk D (or a disk with a larger number of zeros) would have been taken instead of D_{m+1} at the $(m+1)$-th step. This contradiction proves the claim.

Besides, the Langer lemma implies the inequality $|z - z_j| \geqslant \pi \frac{j-(n-1)}{(n-1)}$ (otherwise the zeros $z_1, \dots, z_j$ would lie in a disk of radius strictly less than $\pi \frac{j-(n-1)}{(n-1)}$, and, consequently, in the strip of width $\Delta < 2\pi \frac{j-(n-1)}{(n-1)}$). Thus, we have

$$\frac{|Q(z)|}{\max\{\,|Q(t)| : t \in [-\frac{1}{2}, \frac{1}{2}]\,\}} \geqslant \prod_{j=1}^{2(n-1)} \frac{|z - z_j|}{\max\{\,|t - z_j| : t \in [-\frac{1}{2}, \frac{1}{2}]\,\}} \geqslant \prod_{j=1}^{2(n-1)} \frac{|z - z_j|}{1 + |z - z_j|}$$

$$= \prod_{j=1}^{n-1} \frac{|z - z_j|}{1 + |z - z_j|} \times \prod_{j=1}^{n-1} \frac{|z - z_{n-1+j}|}{1 + |z - z_{n-1+j}|} \geqslant \prod_{j=1}^{n-1} \frac{\frac{j}{n-1} h}{1 + \frac{j}{n-1} h} \times \prod_{j=1}^{n-1} \frac{\frac{\pi j}{n-1}}{1 + \frac{\pi j}{n-1}}$$

$$\geqslant (8h)^{n-1} \times \prod_{j=1}^{n-1} \frac{\frac{j}{n-1}\frac{1}{8}}{1 + \frac{j}{n-1}\frac{1}{8}} \times \prod_{j=1}^{n-1} \frac{\frac{3j}{n-1}}{1 + \frac{3j}{n-1}}.$$

But for each $\theta > 0$ we have

$$\prod_{j=1}^{n-1} \frac{\frac{j}{n-1}\theta}{1 + \frac{j}{n-1}\theta} \geqslant \exp\left\{ (n-1) \int_0^1 \log \frac{\theta t}{1 + \theta t}\, dt \right\} = \left\{ \frac{\theta}{(1+\theta)^{1+1/\theta}} \right\}^{n-1},$$

whence it follows that

$$\frac{|Q(z)|}{\max\{\,|Q(t)| : t \in [-\frac{1}{2}, \frac{1}{2}]\,\}} \geqslant (8h)^{n-1} \left\{ 8 \times \left(\frac{9}{8}\right)^9 \times \frac{4\sqrt[3]{4}}{3} \right\}^{-(n-1)} \geqslant \left\{ \frac{8h}{32\sqrt[3]{4}} \right\}^{n-1}.$$

The measure of the exceptional set $[-\frac{1}{2}, \frac{1}{2}] \cap \left(\bigcup_{k=1}^{s} D'_k\right)$ is at most $8h$. Combining all these estimates, we find ($h \overset{\text{def}}{=} \mu(E)/8$):

$$\sup_{t \in I} |p(t)| \leqslant \left\{ \frac{96\sqrt[3]{4}}{\mu(E)} \right\}^{n-1} \sup_{t \in E} |p(t)| \leqslant \left\{ \frac{154}{\mu(E)} \right\}^{n-1} \sup_{t \in E} |p(t)|.$$

Case 2: $\lambda > n - 1$. We shall reduce this case to Case 1 in the same way as in §3. To do this, we need a weak type estimate of the logarithmic derivative of an exponential polynomial.

Lemma 1.6. *Let* $g(t) = \sum_{k=1}^{n} c_k e^{i\lambda_k t}$ $(c_k \in \mathbb{C},\ 0 = \lambda_1 < \cdots < \lambda_n = \lambda)$. If $\lambda \geqslant n-1$, then $\mu\{\, t \in [-\frac{1}{2}, \frac{1}{2}] : |\frac{d}{dt} \log g(t)| > y \,\} \leqslant 29\lambda/y$ for all $y > 0$.

Assume for the moment that Lemma 1.6 is proved; we can easily finish the proof of Theorem 1.2 by constructing a sequence of exponential polynomials $p_n, p_{n-1}, \ldots, p_s$ $(s \geqslant 1)$ such that

1) $p_n = p$;
2) $\operatorname{ord} p_k = k$ $(k = s, \ldots, n)$;
3) $\|p_{k-1}\|_\infty \geqslant \frac{1}{58}\|p_k\|_\infty$ $(k = s+1, \ldots, n)$;
4) the ratio $\varphi_k \overset{\text{def}}{=} |\frac{p_{k-1}}{p_k}|$ satisfies the weak type estimate $\mu\{\, x \in [-\frac{1}{2}, \frac{1}{2}] : \varphi_k(x) > t \,\} \leqslant \frac{1}{t}$ for $t > 0$;
5) the difference between the greatest and the smallest exponent of p_s does not exceed $s - 1$ (i.e., p_s meets the condition of Case 1 investigated above).

The construction is almost the same as in §3. The difference is that, firstly, we make use of the identity $\underline{q}(t) - \overline{q}(t) = i(\rho_k - \rho_1)p_k(t)$, where

$$\begin{aligned} p_k(t) &\overset{\text{def}}{=} \sum_{m=1}^{k} d_k e^{i\rho_m t} \qquad (\rho_1 < \cdots < \rho_k \in \mathbb{R}),\\ \underline{q}(t) &\overset{\text{def}}{=} e^{i\rho_1 t}\frac{d}{dt}(e^{-i\rho_1 t} p_k(t)),\\ \overline{q}(t) &\overset{\text{def}}{=} e^{i\rho_k t}\frac{d}{dt}(e^{-i\rho_k t} p_k(t)), \end{aligned}$$

to estimate the sum of norms $\|\underline{q}\|_\infty + \|\overline{q}\|_\infty$ from below, and, secondly, we stop the sequence at the polynomial p_s satisfying the condition of Case 1, i.e., at that very moment when we cannot apply Lemma 1.6 to estimate φ_s once more. The same reasoning as in §3 shows that $\left|\frac{p_s(t)}{p_n(t)}\right| \leqslant \left\{\frac{2e}{\mu(E)}\right\}^{n-s}$ outside an exceptional set E' of measure $\mu(E') \leqslant \mu(E)/2$. Thus,

$$\begin{aligned} \left(\frac{1}{58}\right)^{n-s} \|p\|_\infty \leqslant \|p_s\|_\infty &\leqslant \left\{\frac{154}{\mu(E \setminus E')}\right\}^{s-1} \sup_{t \in E \setminus E'} |p_s(t)| \\ &\leqslant \left\{\frac{308}{\mu(E)}\right\}^{s-1} \left\{\frac{2e}{\mu(E)}\right\}^{n-s} \sup_{t \in E} |p(t)|. \end{aligned}$$

Now Theorem 1.5 easily follows if we take into account the inequality $116e < 316$.

It remains to prove Lemma 1.6. We proceed like we did in Case 1, but now the number λ will play the main role instead of $n-1$. Let z_j be the complex zeros of $g(z)$ enumerated in the order of increase of $|\operatorname{Re} z_j|$. The Langer lemma yields $|\operatorname{Re} z_j| \geqslant \pi\frac{j-(n-1)}{(n-1)} \geqslant \frac{\pi}{\lambda}(j - (n-1))$. We again write the Hadamard factorization

$$g(z) = ce^{az} \prod_{j \leqslant 2\lambda} (z - z_j) \prod_{j > 2\lambda} (1 - \frac{z}{z_j})e^{z/z_j} \overset{\text{def}}{=} ce^{az} Q(z) R(z).$$

The estimate for $|\frac{d}{dz}\log R(z)|$. As before, we notice that $|\operatorname{Re} z_j| \geqslant \pi$ if $j > 2\lambda$. Let $|\operatorname{Re} z| < \pi/2$, then

$$\Big|\frac{d}{dz}\log R(z)\Big| \leqslant |z| \sum_{j>2\lambda} \frac{1}{|\operatorname{Re} z_j|(|\operatorname{Re} z_j| - \pi/2)} \leqslant 2|z| \sum_{j>2\lambda} \frac{1}{|\operatorname{Re} z_j|^2}$$
$$\leqslant 2|z| \sum_{j>2\lambda} \frac{\lambda^2}{\pi^2} \frac{1}{(j-(n-1))^2} \leqslant 2\frac{\lambda^2}{\pi^2}|z| \sum_{j>2\lambda} \int_{j-(n-1)-1/2}^{j-(n-1)+1/2} \frac{dt}{t^2}.$$

But if $j > 2\lambda > 2(n-1)$, then $j \geqslant 2n-1$, and $j-(n-1)-1/2 \geqslant 1/2 \geqslant \lambda$. Therefore,

$$\sum_{j>2\lambda} \int_{j-(n-1)-1/2}^{j-(n-1)+1/2} \frac{dt}{t^2} \leqslant \int_\lambda^\infty \frac{dt}{t^2} = \frac{1}{\lambda},$$

and $\left|\frac{d}{dz}\log R(z)\right| \leqslant \frac{2|z|\lambda}{\pi^2}$ if $|\operatorname{Re} z| < \pi/2$. In particular, $\left|\frac{d}{dz}\log R(z)\right| \leqslant \frac{\lambda}{\pi^2}$ on the interval $[-\frac{1}{2}, \frac{1}{2}]$.

The estimate for $|a|$. One may proceed like it was done above to obtain the estimate for $\operatorname{Re} a$, i.e., to consider the argument increment of $g(z)$ along the segment $\left[-i\omega\frac{\overline{a}}{|a|}, i\omega\frac{\overline{a}}{|a|}\right]$, where $0 < \omega < \pi$ (if we let $\omega \to \pi$, the estimate for $R(z)$ fails) but for diversity we go another way and use ... the Turan lemma! Consider an exponential polynomial $\tilde{g}(t) \overset{\text{def}}{=} e^{\lambda t} g\big(-\frac{\overline{a}}{|a|}t\big)$ on the interval $t \in [-\frac{3}{2}, \frac{3}{2}]$; its remarkable property is that the real parts of exponents in its terms are nonnegative. The reasoning of the first half of §1 ensures the estimate

$$\sup_{t\in[-\frac{3}{2},-\frac{1}{2}]} |\tilde{g}(t)| \leqslant \sup_{t\in[-\frac{3}{2},\frac{3}{2}]} |\tilde{g}(t)| \leqslant (12e)^{n-1} \sup_{t\in[\frac{1}{2},\frac{3}{2}]} |\tilde{g}(t)| \leqslant (12e)^{\lambda} \sup_{t\in[\frac{1}{2},\frac{3}{2}]} |\tilde{g}(t)|.$$

The function $Q\big(-\frac{\overline{a}}{|a|}t\big)$ is an algebraic polynomial of degree at most 2λ, consequently, it is a limit of exponential polynomials of order at most $2\lambda+1$ with purely imaginary exponents. Applying the Turan lemma again, we obtain the inequalities

$$\sup_{t\in[-\frac{3}{2},-\frac{1}{2}]} \left|Q\big(-\frac{\overline{a}}{|a|}t\big)\right| \geqslant (12e)^{-2\lambda} \sup_{t\in[-\frac{3}{2},\frac{3}{2}]} \left|Q\big(-\frac{\overline{a}}{|a|}t\big)\right|$$
$$\geqslant (12e)^{-2\lambda} \sup_{t\in[\frac{1}{2},\frac{3}{2}]} \left|Q\big(-\frac{\overline{a}}{|a|}t\big)\right|$$

and

$$\inf_{t\in[-\frac{3}{2},-\frac{1}{2}]} \left|R\big(-\frac{\overline{a}}{|a|}t\big)\right| \geqslant \exp\Big\{-\int_{-3/2}^{3/2} \left|\frac{d}{dt}\log R\big(-\frac{\overline{a}}{|a|}t\big)\right| dt\Big\} \sup_{t\in[-\frac{3}{2},\frac{3}{2}]} \left|R\big(-\frac{\overline{a}}{|a|}t\big)\right|$$
$$\geqslant \exp\Big\{-\int_{-3/2}^{3/2} \frac{2|t|\lambda}{\pi^2}\, dt\Big\} \sup_{t\in[\frac{1}{2},\frac{3}{2}]} \left|R\big(-\frac{\overline{a}}{|a|}t\big)\right|$$
$$\geqslant e^{-\lambda/2} \sup_{t\in[\frac{1}{2},\frac{3}{2}]} \left|R\big(-\frac{\overline{a}}{|a|}t\big)\right|.$$

If $|a| > \lambda$, then $\inf_{t\in[-\frac{3}{2},-\frac{1}{2}]} |ce^{(\lambda-|a|)t}| \geqslant e^{|a|-\lambda} \sup_{t\in[\frac{1}{2},\frac{3}{2}]} |ce^{(\lambda-|a|)t}|$. Since $\tilde{g}(t) = ce^{(\lambda-|a|)t}Q\big(-\frac{\overline{a}}{|a|}t\big)R\big(-\frac{\overline{a}}{|a|}t\big)$, these inequalities may hold simultaneously only if $|a| \leqslant \lambda(3\log(12e) + 1/2 + 1) \leqslant \frac{25}{2}\lambda$.

The polynomial $Q(z)$. By Lemma 1.2, the polynomial $Q(z)$ satisfies the weak type estimate $\mu\big\{\, t \in [-\frac{1}{2}, \frac{1}{2}] : \big|\frac{d}{dt}\log Q(t)\big| > y \,\big\} \leqslant \frac{16\lambda}{y}$ on the segment $[-\frac{1}{2}, \frac{1}{2}]$.

If we combine all the above estimates and make use of the inequality $\big|\frac{d}{dt}\log g(t)\big| \leqslant |a| + \big|\frac{d}{dt}\log R(t)\big| + \big|\frac{d}{dt}\log Q(t)\big|$, we obtain

$$\begin{aligned}
&\mu\Big\{\, t \in [-\frac{1}{2}, \frac{1}{2}] : \big|\frac{d}{dt}\log g(t)\big| > y \,\Big\} \\
&\qquad \leqslant \mu\Big\{\, t \in [-\frac{1}{2}, \frac{1}{2}] : \big|\frac{d}{dt}\log Q(t)\big| > y - 13\lambda \,\Big\} \\
&\qquad \leqslant \frac{16\lambda}{y - 13\lambda} \leqslant \frac{29\lambda}{y}
\end{aligned}$$

for $y \geqslant 29\lambda$. But if $y < 29\lambda$, then the corresponding estimate becomes trivial because $\frac{29\lambda}{y} \geqslant 1 = \mu([-\frac{1}{2}, \frac{1}{2}])$. Lemma 1.6 is proved.

C hapter 2. Random periodization technique and the Morgan theorem

In this chapter we are going to present a very simple method (borrowed from geometry of numbers and, in the main, going back to Siegel), which will allow us to reduce the study of a function $f \in L^2(\mathbb{R})$ with spectrum of finite measure to the study of some family of trigonometric polynomials with uniformly bounded orders on the unit circumference $\mathbb{T}$. In this way the following inequality of "uncertainty principle type" will be derived from the Turan lemma:

$$\|f\|^2_{L^2(\mathbb{R})} \leqslant Ae^{A\mu(E)\mu(\Sigma)}\Big(\int_{\mathbb{R}\smallsetminus E}|f|^2 + \int_{\mathbb{R}\smallsetminus\Sigma}|\widehat{f}|^2\Big); \tag{2.1}$$

here E and Σ are arbitrary sets of finite Lebesgue measure (and A is an absolute constant). This inequality essentially improves the famous Amrein–Berthier theorem. An immediate consequence of (2.1) is the uniqueness Theorem 2.3, resembling the Beurling theorem on the divergence of the double integral

$$\iint_{\mathbb{R}^2} |f(x)||\widehat{f}(y)|e^{2\pi|x||y|}\,dx\,dy$$

for every function $f \in L^2(\mathbb{R})$ not vanishing identically, and, in a sense, generalizing the latter. In Theorems 2.4 and 2.6 we try to treat the case where we have some information about "smallness" of $|f|$ and $|\widehat{f}|$ not on the whole real line $\mathbb{R}$ but only on one of its half-lines, say, $\mathbb{R}_-$ (though we do not succeed very much). We postpone the discussion of the corresponding classical results to §3.

§2.1. Lattice averaging lemma and random periodization operators

The following simple lemma will be very useful for us.

Lemma 2.1 (the lattice averaging lemma). *Let* $\varphi\colon \mathbb{R} \to \mathbb{R}_+$ *be a positive summable function*, and let $\varepsilon > 0$ be fixed. Then

$$\int_1^2 \sum_{k\in\mathbb{Z}\setminus\{0\}} \varphi(k\varepsilon v)\,dv \leqslant \frac{1}{\varepsilon}\int_{\mathbb{R}} \varphi(t)\,dt$$

and

$$\int_1^2 \sum_{k\in\mathbb{Z}\setminus\{0\}} \varphi\left(\frac{k}{\varepsilon v}\right) dv \leqslant 4\varepsilon \int_{\mathbb{R}} \varphi(t)\,dt.$$

Proof. We have:

$$\begin{aligned}\int_1^2 \sum_{k>0}\varphi(k\varepsilon v)\,dv &= \sum_{k>0}\int_1^2 \varphi(k\varepsilon v)\,dv \overset{(t=k\varepsilon v)}{=} \sum_{k>0}\frac{1}{\varepsilon k}\int_{\varepsilon k}^{2\varepsilon k}\varphi(t)\,dt\\ &= \frac{1}{\varepsilon}\sum\int_{\substack{\varepsilon k<t<2\varepsilon k\\ k>0}} \frac{\varphi(t)}{k}\,dt = \frac{1}{\varepsilon}\int_0^{+\infty}\varphi(t)\Bigg(\sum_{\frac{t}{2\varepsilon}<k<\frac{t}{\varepsilon}}\frac{1}{k}\Bigg)dt \leqslant \frac{1}{\varepsilon}\int_0^{+\infty}\varphi(t)\,dt,\end{aligned}$$

since $\sum_{\frac{t}{2\varepsilon}<k<\frac{t}{\varepsilon}} \frac{1}{k} \leqslant 1$ for $t > 0$.

Similarly,

$$\int_1^2 \sum_{k>0}\varphi(k\varepsilon v)\,dv \leqslant \frac{1}{\varepsilon}\int_{-\infty}^0 \varphi(t)\,dt.$$

Summing these inequalities, we obtain the first statement of the lemma. Further,

$$\begin{aligned}\int_1^2 \sum_{k>0}\varphi\left(\frac{k}{\varepsilon v}\right) dv &= \sum_{k>0}\int_1^2 \varphi\left(\frac{k}{\varepsilon v}\right) dv = \sum_{k>0}\frac{k}{\varepsilon}\int_{k/(2\varepsilon)}^{k/\varepsilon}\frac{\varphi(t)}{t^2}\,dt = \frac{1}{\varepsilon}\sum\int_{\substack{\frac{k}{2\varepsilon}<t<\frac{k}{\varepsilon}\\ k>0}}\frac{\varphi(t)k}{t^2}\,dt\\ &= \frac{1}{\varepsilon}\int_0^{+\infty}\varphi(t)\Bigg(\sum_{\varepsilon t<k<2\varepsilon t} k\Bigg)\frac{dt}{t^2} \leqslant 4\varepsilon\int_0^{+\infty}\varphi(t)\,dt,\end{aligned}$$

since $\sum_{\varepsilon t<k<2\varepsilon t} k \leqslant 4\varepsilon^2 t^2$ for $t > 0$. The rest is clear.

Now let $E \subset \mathbb{R}$ be a measurable set of finite measure. We consider an arbitrary function $f \in L^2(\mathbb{R})$ supported on E and fix a positive number ε. We define the random periodization g of the function f by

$$g(t) = g(\varepsilon, v|t) \overset{\text{def}}{=} \frac{1}{\sqrt{\varepsilon v}}\sum_{k\in\mathbb{Z}} f\left(\frac{k+t}{\varepsilon v}\right).$$

Here v is a random variable equidistributed on the interval (1,2). The series in the definition of g converges in $L^1_{\mathrm{loc}}(\mathbb{R})$; for every v this series represents a 1-periodic function. An easy computation shows that the Fourier coefficients of g are $\hat{g}_m = \sqrt{\varepsilon v}\widehat{f}(m\varepsilon v)$ $(m \in \mathbb{Z})$. ¿From the lattice averaging lemma we shall derive several simple but useful properties of the random periodization.

Proposition 2.2.

(A) $\mu\{\, t \in (0,1) : g(t) \neq 0 \,\} \leqslant 2\varepsilon\mu(E)$;

(B) $\mathbf{E}\|g\|^2_{L^2(0,1)} \leqslant 2\varepsilon|\widehat{f}(0)|^2 + 2\|f\|^2_{L^2(\mathbb{R})} \leqslant 2(\varepsilon\mu(E)+1)\|f\|^2_{L^2(\mathbb{R})}$.

Let $\Sigma \subset \mathbb{R}$ be measurable, $0 \in \Sigma$. We consider a random lattice $\Lambda = \Lambda(\varepsilon, v) \overset{\text{def}}{=} \{\, s\varepsilon v : s \in \mathbb{Z} \,\}$ and denote $\mathfrak{M} = \{\, s \in \mathbb{Z} : s\varepsilon v \in \Sigma \,\}$.

(C) $\mathbf{E}(\operatorname{card}\mathfrak{M} - 1) \leqslant \frac{\mu(\Sigma)}{\varepsilon}$;

(D) $\mathbf{E}\sum_{m \in \mathbb{Z}\smallsetminus\mathfrak{M}} |\hat{g}_m|^2 \leqslant 2\int_{\mathbb{R}\smallsetminus\Sigma} |\widehat{f}|^2$.

Proof. (A): The measure of the set of all points $t \in (0,1)$ for which the summand $f(\frac{k+t}{\varepsilon v})$ in the series defining g does not vanish is equal to $\mu(\varepsilon v E \cap (k, k+1))$. Therefore,

$$\mu\{\, t \in (0,1) : g(t) \neq 0 \,\} \leqslant \sum_{k \in \mathbb{Z}} \mu(\varepsilon v E \cap (k, k+1)) = \mu(\varepsilon v E) \leqslant 2\varepsilon\mu(E).$$

(B):

$$\mathbf{E}\|g\|^2_{L^2(0,1)} = \mathbf{E}\sum_{k \in \mathbb{Z}} |\hat{g}_k|^2 = \mathbf{E}|\hat{g}_0|^2 + \mathbf{E}\sum_{k \in \mathbb{Z}\smallsetminus\{0\}} |\hat{g}_k|^2.$$

But $|\hat{g}_0|^2 = \varepsilon v|\widehat{f}(0)|^2 \leqslant 2\varepsilon|\widehat{f}(0)|^2$, and

$$\begin{aligned}\mathbf{E}\sum_{k \in \mathbb{Z}\smallsetminus\{0\}} |\hat{g}_k|^2 &= \int_1^2 \Bigl(\sum_{k \in \mathbb{Z}\smallsetminus\{0\}} \varepsilon v|\widehat{f}(k\varepsilon v)|^2\Bigr)\, dv \leqslant 2\varepsilon\int_1^2 \Bigl(\sum_{k \in \mathbb{Z}\smallsetminus\{0\}} |\widehat{f}(k\varepsilon v)|^2\Bigr)\, dv \\ &\leqslant 2\int_{\mathbb{R}} |\widehat{f}|^2 = 2\|f\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

It remains to notice that

$$|\widehat{f}(0)|^2 = \left|\int_E f\right|^2 \leqslant \mu(E)\int_E |f|^2 = \mu(E)\|f\|^2_{L^2(\mathbb{R})}.$$

(C): Since $\operatorname{card}\mathfrak{M} = 1 + \sum_{k \in \mathbb{Z}\smallsetminus\{0\}} \chi_\Sigma(k\varepsilon v)$, we have

$$\mathbf{E}(\operatorname{card}\mathfrak{M} - 1) = \int_1^2 \sum_{k \in \mathbb{Z}\smallsetminus\{0\}} \chi_\Sigma(k\varepsilon v)\, dv \leqslant \frac{1}{\varepsilon}\int_{\mathbb{R}} \chi_\Sigma = \frac{\mu(\Sigma)}{\varepsilon}.$$

(D):

$$\begin{aligned}\mathbf{E}\sum_{m \in \mathbb{Z}\smallsetminus\mathfrak{M}} |\hat{g}_m|^2 &= \int_1^2 \Bigl(\sum_{k \in \mathbb{Z}\smallsetminus\{0\}} \varepsilon v|\widehat{f}(k\varepsilon v)|^2 \chi_{\mathbb{R}\smallsetminus\Sigma}(k\varepsilon v)\Bigr)\, dv \\ &= 2\varepsilon\int_1^2 \Bigl(\sum_{k \in \mathbb{Z}\smallsetminus\{0\}} \bigl(|\widehat{f}(k\varepsilon v)|^2 \chi_{\mathbb{R}\smallsetminus\Sigma}\bigr)(k\varepsilon v)\Bigr)\, dv \\ &\leqslant 2\int_{\mathbb{R}} |\widehat{f}|^2 \chi_{\mathbb{R}\smallsetminus\Sigma} = 2\int_{\mathbb{R}\smallsetminus\Sigma} |\widehat{f}|^2. \quad \text{(By L.A.L.)}\end{aligned}$$

§2.2. Strong annihilation of sets of finite measure

Let E and Σ be two measurable subsets of $\mathbb{R}$. Borrowing the terminology from Jöricke and Havin, we say that E and Σ annihilate if for every function $f \in L^2(\mathbb{R})$ the conditions $\operatorname{supp} f \subset E$, $\operatorname{spec} f \subset \Sigma$ imply that f vanishes identically. We say that E and Σ strongly annihilate if there exists a constant $C > 0$ such that the inequality

$$\|f\|^2_{L^2(\mathbb{R})} \leqslant C\Big(\int_{\mathbb{R}\setminus E} |f|^2 + \int_{\mathbb{R}\setminus\Sigma} |\widehat{f}|^2\Big) \tag{$*$}$$

holds for every function $f \in L^2(\mathbb{R})$. The strong annihilation condition can be written in a form which is less symmetric but more convenient to verify: E and Σ strongly annihilate if and only if

$$\int_{\Sigma} |\widehat{f}|^2 \leqslant C' \int_{\mathbb{R}\setminus\Sigma} |\widehat{f}|^2 \tag{$**$}$$

for every $f \in L^2(\mathbb{R})$ supported on E.

There is a relationship between the best possible constants C and C':

$$C' = \operatorname{ctg}^2\alpha, \qquad C = \frac{1}{2\sin^2\frac{\alpha}{2}} = \frac{1}{1-\cos\alpha},$$

where α is the angle between the subspaces $L^2(E) \overset{\text{def}}{=} \{\, f \in L^2(\mathbb{R}) : \operatorname{supp} f \subset E \,\}$ and $L^2(\widehat{\Sigma}) \overset{\text{def}}{=} \{\, f \in L^2(\mathbb{R}) : \operatorname{spec} f \subset \Sigma \,\}$ of the Hilbert space $L^2(\mathbb{R})$. The proof of this statement is a simple exercise in geometry. Denoting by P_E and $P_{\widehat{\Sigma}}$ the orthogonal projection onto $L^2(E)$ and $L^2(\widehat{\Sigma})$, respectively, we have:

$$\begin{aligned}\cos\alpha &= \sup\{\, |(f,g)| : f \in L^2(E),\ g \in L^2(\widehat{\Sigma}),\ \|f\|^2_{L^2(\mathbb{R})} = \|g\|^2_{L^2(\mathbb{R})} = 1 \,\}\\ &= \sup\{\, |(P_{\widehat{\Sigma}} f, g)| : \dots \,\}\\ &= \sup\{\, \|P_{\widehat{\Sigma}} f\|_{L^2(\mathbb{R})} : f \in L^2(E),\ \|f\|^2_{L^2(\mathbb{R})} = 1 \,\},\end{aligned}$$

and

$$\begin{aligned}C' &= \sup\left\{ \frac{\int_\Sigma |\widehat{f}|^2}{\int_{\mathbb{R}\setminus\Sigma} |\widehat{f}|^2} : f \in L^2(E),\ \|f\|^2_{L^2(\mathbb{R})} = 1 \right\}\\ &= \sup\left\{ \frac{\|P_{\widehat{\Sigma}} f\|^2_{L^2(\mathbb{R})}}{1 - \|P_{\widehat{\Sigma}} f\|^2_{L^2(\mathbb{R})}} : f \in L^2(E),\ \|f\|^2_{L^2(\mathbb{R})} = 1 \right\}\\ &= \frac{\cos^2\alpha}{1-\cos^2\alpha} = \operatorname{ctg}^2\alpha.\end{aligned}$$

The computation of the constant C is slightly more complicated. Denote by β and γ the angles between f and the subspaces $L^2(E)$ and $L^2(\widehat{\Sigma})$, respectively. It is clear that $0 < \beta, \gamma < \frac{\pi}{2}$, $\beta + \gamma \geqslant \alpha$. Since

$$\begin{aligned}\int_{\mathbb{R}\setminus E} |f|^2 + \int_{\mathbb{R}\setminus\Sigma} |\widehat{f}|^2 &= \|f - P_E f\|^2_{L^2(\mathbb{R})} + \|f - P_{\widehat{\Sigma}} f\|^2_{L^2(\mathbb{R})}\\ &= (\sin^2\beta + \sin^2\gamma)\|f\|^2_{L^2(\mathbb{R})} \geqslant 2\sin^2\frac{\alpha}{2}\|f\|^2_{L^2(\mathbb{R})},\end{aligned}$$

we have $C \leqslant \frac{1}{2\sin^2\frac{\alpha}{2}}$. To verify the reverse inequality, it suffices to exhibit a function f for which the angles β and γ are close to $\frac{\alpha}{2}$. This can be done as follows. One can choose $g \in L^2(E)$ and $h \in L^2(\widehat{\Sigma})$ so that $\|g\|_{L^2(\mathbb{R})} = \|h\|_{L^2(\mathbb{R})} = 1$ and $\operatorname{Re}(h, g) \approx \cos\alpha$, and then put $f \overset{\text{def}}{=} \frac{1}{2}(g+h)$.

It should be noted that, proceeding in the same way, one can describe the image of the unit ball of $L^2(\mathbb{R})$ under the mapping

$$L^2(\mathbb{R}) \ni f \to \left(\int_{\mathbb{R}\smallsetminus E} |f|^2, \int_{\mathbb{R}\smallsetminus\Sigma} |\widehat{f}|^2\right) \in \mathbb{R}^2_+$$

provided that each of the subspaces $L^2(E)$ and $L^2(\widehat{\Sigma})$ contains a vector making an angle arbitrarily close to $\frac{\pi}{2}$ with the other subspace (this condition is certainly satisfied if both E and Σ have zero density at infinity, i.e,. if $\lim_{A\to+\infty}\frac{\mu(E\cap[-A,A])}{A} = \lim_{A\to+\infty}\frac{\mu(\Sigma\cap[-A,A])}{A} = 0$; the corresponding vectors can be chosen among those of the form $fe^{i\lambda t}$ and $\tau_\lambda g$, where $f \in L^2(E)$, $g \in L^2(\widehat{\Sigma})$ and λ is a suitable number from a sufficiently large interval centered at 0). This image turns out to be the square $[0,1]^2$ with the upper-right angle cut off along the curve $\arccos\sqrt{x} + \arccos\sqrt{y} = \alpha$ (this simple observation is due to Slepian and Pollac).

Excluding α from the formulas for C and C', we get

$$C = C' + 1 + \sqrt{C'(C'+1)} \leqslant 2C' + \frac{3}{2}.$$

Now we state the main theorem of this section.

Theorem 2.2. *For every two sets E and Σ of finite Lebesgue measure and every function $f \in L^2(\mathbb{R})$*, the following inequality holds:

$$\|f\|^2_{L^2(\mathbb{R})} \leqslant 130 e^{66\mu(E)\mu(\Sigma)}\left(\int_{\mathbb{R}\smallsetminus E}|f|^2 + \int_{\mathbb{R}\smallsetminus\Sigma}|\widehat{f}|^2\right).$$

Proof. As it was shown above, it suffices to prove that

$$\int_\Sigma |\widehat{f}|^2 \leqslant 64 e^{66\mu(E)\mu(\Sigma)}\int_{\mathbb{R}\smallsetminus\Sigma}|\widehat{f}|^2$$

for every function $f \in L^2(E)$. We set $\varepsilon = \frac{1}{4\mu(E)}$ and introduce the random periodization g of the function f. By (A),

$$\mu\{\, t \in (0,1) : g(t) = 0 \,\} \overset{\text{def}}{=} \mu(F) \geqslant 1 - 2\varepsilon\mu(E) = \frac{1}{2}.$$

We decompose g into a sum $p+q$, where

$$p(t) \overset{\text{def}}{=} \sum_{m: m\varepsilon v\in\Sigma\cup\{0\}} \hat{g}_m e^{2\pi imt} \overset{\text{def}}{=} \sum_{m\in\mathfrak{M}} \hat{g}_m e^{2\pi imt}$$

(the second relation defines $\mathfrak{M}$) and $q(t) \stackrel{\text{def}}{=} \sum_{m\in\mathbb{Z}\setminus\mathfrak{M}} \hat{g}_m e^{2\pi imt}$. We have

$$\mathbf{E}\|q\|^2_{L^2(0,1)} = \mathbf{E}\sum_{m\in\mathbb{Z}\setminus\mathfrak{M}} |\hat{g}_m|^2 \stackrel{\text{(D)}}{\leqslant} 2\int_{\mathbb{R}\setminus\Sigma} |\widehat{f}|^2,$$

whence

$$\mathbf{P}\left\{\|q\|^2_{L^2(0,1)} > 4\int_{\mathbb{R}\setminus\Sigma} |\widehat{f}|^2\right\} < \frac{1}{2}.$$

Next,

$$\mathbf{E}(\operatorname{ord} p - 1) = \mathbf{E}(\operatorname{card}\mathfrak{M} - 1) \stackrel{\text{(C)}}{\leqslant} \frac{\mu(\Sigma)}{\varepsilon} = 4\mu(E)\mu(\Sigma).$$

Consequently,

$$\mathbf{P}\{\operatorname{ord} p > 1 + 8\mu(E)\mu(\Sigma)\} < \frac{1}{2}.$$

We see that,with positive probability, the following 4 events take place simultaneously:

(1) $\mu(F) \geqslant \frac{1}{2}$;
(2) $\|q\|^2_{L^2(0,1)} \leqslant 4\int_{\mathbb{R}\setminus\Sigma} |\widehat{f}|^2$;
(3) $\operatorname{ord} p \leqslant 1 + 8\mu(E)\mu(\Sigma)$;
(4) $\varepsilon|\widehat{f}(0)|^2 = \frac{1}{4\mu(E)}|\widehat{f}(0)|^2 \leqslant |\hat{p}_0|^2 = |\hat{g}_0|^2$

(indeed, (1) and (4) always hold, while each of (2) and (3) does not hold with probability less than $\frac{1}{2}$). Since $g\big|_F \equiv 0$, we have $p\big|_F \equiv -q\big|_F$ and $\int_F |p|^2 = \int_F |q|^2$. Hence $\mu\Big\{ t \in F : |p(t)|^2 \geqslant 16\int_{\mathbb{R}\setminus\Sigma} |\widehat{f}|^2 \Big\} \leqslant \frac{1}{4}$, and, since $\mu(F) \geqslant \frac{1}{2}$, we get

$$\mu\left\{ t \in (0,1) : |p(t)| \leqslant 4\left(\int_{\mathbb{R}\setminus\Sigma} |\widehat{f}|^2\right)^{1/2} \right\} \geqslant \frac{1}{4}.$$

Now a special case of the Turan lemma (Theorem 1.4) implies

$$\begin{aligned}\frac{1}{4\mu(E)}|\widehat{f}(0)|^2 \leqslant |\hat{p}_0|^2 \leqslant \Big(\sum_k |\hat{p}_k|\Big)^2 &\leqslant \left\{\left(\frac{14}{1/4}\right)^{\operatorname{ord} p - 1} 4\left(\int_{\mathbb{R}\setminus\Sigma} |\widehat{f}|^2\right)^{1/2}\right\}^2 \\ &\leqslant 16 \times 56^{16\mu(E)\mu(\Sigma)} \int_{\mathbb{R}\setminus\Sigma} |\widehat{f}|^2,\end{aligned}$$

whence

$$|\widehat{f}(0)|^2 \leqslant 64\mu(E) e^{16\log 56\mu(E)\mu(\Sigma)} \int_{\mathbb{R}\setminus\Sigma} |\widehat{f}|^2.$$

If we take the function $f_1(x) \stackrel{\text{def}}{=} f(x)e^{-2\pi ixy}$ instead of $f(x)$ and the set $\Sigma - y$ instead of Σ, we arrive at the same estimate for $|\widehat{f}(y)|$ and, since y is arbitrary, for $\sup_{\mathbb{R}} |\widehat{f}|$. Integrating this estimate over Σ, we get the inequality

$$\int_\Sigma |\widehat{f}|^2 \leqslant 64\mu(E)\mu(\Sigma)e^{16\log 56\mu(E)\mu(\Sigma)} \int_{\mathbb{R}\setminus\Sigma} |\widehat{f}|^2 \leqslant 64 e^{66\mu(E)\mu(\Sigma)} \int_{\mathbb{R}\setminus\Sigma} |\widehat{f}|^2,$$

which proves the theorem.

Of course, the constant factor in the exponent is estimated very roughly. Analyzing the proof thoroughly one may observe that our method provides a possibility to estimate the constants C and C' from above by $A_\delta e^{(w+\delta)\mu(E)\mu(\Sigma)}$, where $\delta > 0$ is an arbitrary small positive number, A_δ is a constant depending on δ only, and $w \overset{\text{def}}{=} \inf_{\alpha\in(0,1)}\bigl(\frac{2}{\alpha}\log\frac{16e}{\pi}\frac{1}{1-\alpha}\bigr) < 10.6$, which is nevertheless very far from the conjectured inequality $C \leqslant A_\delta e^{(\frac{\pi}{2}+\delta)\mu(E)\mu(\Sigma)}$ (the latter cannot be essentially improved, which can be shown by the example $f = e^{-\pi x^2}$, $E = \Sigma = [-a, a]$, $a \to +\infty$).

Theorem 2.2 is substantial only if $\mu(E)\mu(\Sigma) \geqslant 1$. If $\mu(E)\mu(\Sigma) < 1$, then much simpler ideas work: for every function $f \in L^2(\mathbb{R})$ supported on E, we have $\sup|\widehat{f}| \leqslant \|f\|_{L^1(\mathbb{R})} \leqslant \sqrt{\mu(E)}\|f\|_{L^2(\mathbb{R})}$, whence $\int_\Sigma|\widehat{f}|^2 \leqslant \mu(E)\mu(\Sigma)\|f\|^2_{L^2(\mathbb{R})}$. Consequently, $\int_{\mathbb{R}\setminus\Sigma}|\widehat{f}|^2 \geqslant (1 - \mu(E)\mu(\Sigma))\|f\|^2_{L^2(\mathbb{R})}$ and $\int_\Sigma|\widehat{f}|^2 \leqslant \frac{\mu(E)\mu(\Sigma)}{1-\mu(E)\mu(\Sigma)}\int_{\mathbb{R}\setminus\Sigma}|\widehat{f}|^2$. Recalling the relationship between C and C', we get

$$\|f\|_{L^2(\mathbb{R})} \leqslant \frac{1}{\sqrt{1-\mu(E)\mu(\Sigma)}}\left(\int_{\mathbb{R}\setminus E}|f|^2 + \int_{\mathbb{R}\setminus\Sigma}|\widehat{f}|^2\right)$$

for every $f \in L^2(\mathbb{R})$.

Any nontrivial estimate of the constant C implies some restrictions on the simultaneous distribution of the absolute values of the functions f and $\widehat{f}$. We say that a decreasing function $\Phi\colon (0,+\infty) \to [0,+\infty]$ majorizes $|f|$ in distribution if

$$\mu\{\, t \in \mathbb{R} : |f(t)| > y \,\} \leqslant \mu\{\, t \in (0,+\infty) : \Phi(t) > y \,\}$$

for all $y > 0$. This condition is equivalent to the inequality

$$\mu\{\, t \in \mathbb{R} : |f(t)| > \Phi(y) \,\} \leqslant y$$

for all $y > 0$.

Let Φ and Ψ be majorize $|f|$ and $|hf|$ in distributions. Theorem 2.2 immediately implies the following statement.

Corollary 2.3. *For every $a, b > 0$,*

$$\int_a^{+\infty}\Phi^2 + \int_b^{+\infty}\Psi^2 \geqslant \frac{1}{130}e^{-66ab}\|f\|^2_{L^2(\mathbb{R})}.$$

In particular,

$$\varliminf_{a,b\to+\infty} e^{66ab}\left(\int_a^{+\infty}\Phi^2 + \int_b^{+\infty}\Psi^2\right) = 0$$

implies $f \equiv 0$.

It follows that the majorants Φ and Ψ cannot decrease as e^{-Bx^p} and $e^{-B'x^{p'}}$, respectively, $1/p + 1/p' = 1$ and $B^{1/p}B'^{1/p'} > 33$. The last statement can be regarded as an interesting complement to the Morgan theorem discussed in the next section.

§2.3. The Morgan theorem

The following theorem was proved by Morgan [9] in 1934.

Theorem. *Let $p \in [2, +\infty)$, $p' = \frac{p}{p-1} \in (1, 2]$, $f \in S$.*

(A) If $f(x) = O(e^{-B|x|^p})$, $\widehat{f}(x) = O(e^{-B'|x|^{p'}})$ as $x \to \pm\infty$, and $B^{1/p}B'^{1/p'} > b_1(p) = \frac{2\pi}{p^{1/p}p'^{1/p'}}|\cos\frac{\pi p'}{2}|^{1/p'}$, then $f \equiv 0$.

(B) If $f(x) = O(e^{-B|x|^p})$, $\widehat{f}(x) = O(e^{-B'|x|^{p'}})$ as $x \to -\infty$, and $B^{1/p}B'^{1/p'} > b_2(p) = \frac{2\pi}{p^{1/p}p'^{1/p'}}\sin\frac{\pi}{p'}$, then $f \equiv 0$.

Constants $b_1(p)$ and $b_2(p)$ are the best possible. The conclusion of (A) remains valid if f rapidly decreases only on the left semiaxis while in case (B) an example of a nontrivial function f on which the value $b_2(p)$ is "attained" can be chosen so that $f(-x) = \overline{f(x)}$; the latter means that it does not matter whether the decrease of f is "one-sided" or "two-sided", and the situation is determined by the character of decrease of $\widehat{f}$ (the functions f and $\widehat{f}$ do not have the same rights because p and p' do not).

We make several remarks concerning the Morgan theorem before proving it.

1) The statement of the theorem is invariant with respect to the linear change of variable $x \to \alpha x$ ($\alpha > 0$). Substituting $\overline{f}$ in place of f, we can see that it does not matter on which particular semiaxis $\widehat{f}$ rapidly decreases.

2) The condition $f \in S$ may be replaced by the condition of belonging to an arbitrary other reasonable function class between S and S'. To see this, it suffices to convolve f with a boundedly supported infinitely differentiable function and then to do the same with $\widehat{f}$.

3) Moreover, the convolution allows us to derive (B) from (A) provided we do not aim at making the constant $b_2(p)$ best possible. To do this, one can consider the function $g \overset{\text{def}}{=} (f\overline{\widetilde{f}}) * (\overline{f}\widetilde{f})$ (under the assumptions of (B)), where $\widetilde{f}(x) \overset{\text{def}}{=} f(-x)$. Easy estimates show that g decreases on each of the two semiaxis as $e^{-B(\frac{|x|}{2})^p}$, while $\hat{g}$ decreases as $e^{-B'(\frac{|x|}{2})^{p'}}$, and, therefore, one can take $b_2(p) = 4b_1(p)$.

4) The Paley–Wiener theorem implies that the one-sided condition $f(x) = O(e^{-\frac{(2\pi+\delta)|x|^p}{p}})$, $x \to -\infty$, is equivalent to the analytic continuability of the function $\widehat{f}$ into the upper half-plane $\{\operatorname{Im} z \geqslant 0\}$ with the growth estimate $\widehat{f}(z) = O\big(\exp\{(2\pi - \delta)\frac{|\operatorname{Im} z|^{p'}}{p'}\}\big)$, and the two-sided condition $f(x) = O(e^{-\frac{(2\pi+\delta)|x|^p}{p}})$, $x \to \pm\infty$, means that $\widehat{f}$ can be extended to an entire function with the same growth estimate (provided $f \in S$, of course). Consequently (the details are left to the reader), the Morgan theorem is equivalent to the following statement.

Let $q \in (1, 2]$, and let $F(z)$ be a function analytic in the upper half-plane, continuous up to its boundary and satisfying the conditions $F\big|_{\mathbb{R}} \in S$ and $F(z) = O(e^{(1-\varepsilon)|\operatorname{Im} z|^q})$ with some $\varepsilon \in (0, 1)$.

(A) If $F(x) = O(e^{-B|x|^q})$ as $x \to \pm\infty$, $x \in \mathbb{R}$, and $B > |\cos\frac{\pi q}{2}|$, then $F \equiv 0$.

(B) If $F(x) = O(e^{-B|x|^q})$ as $x \to -\infty$, $x \in \mathbb{R}$, and $B > (\sin\frac{\pi}{2})^q$, then $F \equiv 0$.

The constants $|\cos\frac{\pi q}{2}|$ and $(\sin\frac{\pi}{2})^q$ cannot be improved even for entire functions satisfying the same growth condition.

Proof of the Morgan theorem. (A) We set $G(z) = e^{-(\frac{z}{i})^q}F(z)$ (choosing the branch of the q-th power to be positive on $(0, +\infty)$). $G(z)$ is bounded in a small angle bisected by the imaginary axis. Therefore, $G \in H^\infty(\mathbb{C}_+)$ by the Fragmen–Lindelöff principle (the latter can be applied since $q \leqslant 2$). But $\log|G(x)|$ behaves on the real line like $-(B+\cos\frac{\pi q}{2})|x|^q$, whence $\int_{\mathbb{R}} \log|G(x)|\frac{dx}{1+x^2} = -\infty$ provided $B > |\cos\frac{\pi q}{2}|$. This implies $G \equiv 0$ and $F \equiv 0$.

(B) We set $G(z) = e^{(z\sin\frac{\pi}{q})^q}F(z)$. This function is bounded on $(0,+\infty)$ and in a small angle bisected by the ray $\Gamma_{\frac{\pi}{q}}$ ($\Gamma_\alpha \overset{\text{def}}{=} \{\, e^{i\alpha}t : t \geqslant 0\,\}$). Therefore, of the Fragmen–Lindelöff principle, G is bounded inside the angle of angular measure $\frac{\pi}{q}$ bounded by the rays $[0,+\infty) = \Gamma_0$ and $\Gamma_{\frac{\pi}{q}}$. But on these rays $G(z)$ behaves like $e^{-\delta|z|^q}$, and, therefore, the integral of $\log|G|$ over the boundary of the angle with respect to the harmonic measure diverges, whence $G \equiv 0$ and $F \equiv 0$.

Now we shall show how to construct examples of symmetric (with respect to $\mathbb{R}$) entire functions (i.e., those satisfying $F(\overline{z}) = \overline{F(z)}$) with almost optimal correlation between the rate of growth and decrease. If $q = 2$, then $F(z) = e^{-z^2}$ is an example for whatever we could imagine. Let $q \in (1,2)$. Consider the canonical product $P(z) = \prod_{n=1}^{\infty}(1+\frac{z}{n^{1/q}})\exp(-\frac{z}{n^{1/q}})$. The function $\log|F(z)|$ can be estimated from above by the integral

$$\operatorname{Re}\int_0^{+\infty}\Big(\log\Big(1+\frac{z}{t^{1/q}}\Big)-\frac{z}{t^{1/q}}\Big)\,dt = \operatorname{Re}\Big(z^q\int_{\Gamma_{-q\arg z}}\Big(\log\Big(1+\frac{1}{s^{1/q}}\Big)-\frac{1}{s^{1/q}}\Big)\,ds\Big)$$
$$= \quad -\gamma\operatorname{Re} z^q = -\gamma|z|^q\cos(q\arg z),$$

up to a summand of order $o(|z|^q)$ depending only on $|z|$ and not depending on $\arg z \in [-\pi,\pi]$.

(A) The absolute value of the function $F(z) = P(z)P(-z)$ does not exceed $\exp\{-\gamma(\cos q\theta + \cos q(\pi-\theta))R^q + o(R^q)\} = \exp\{-2\gamma\cos\frac{\pi q}{2}\cos q(\theta-\frac{\pi}{2})R^q + o(R^q)\}$ on the rays $Re^{\pm i\theta}$ ($\theta \in [0,\pi]$). The ratio $\frac{\cos q(\theta-\frac{\pi}{2})}{(\sin\theta)^q}$ attains its maximum when $\theta = \frac{\pi}{2}$; therefore, $|F(z)| \leqslant \exp\{2\gamma|\cos\frac{\pi q}{2}||\operatorname{Im} z|^q + o(|\operatorname{Im} z|^q)\}$ (we can replace $o(|z|^q)$ by $o(|\operatorname{Im} z|^q)$ because $\cos q(\theta-\frac{\pi}{2})$ is negative if θ is near to 0 or π, whence it follows that F is a rapidly decreasing function in a small angle bisected by the real axis). On the other hand, $|F(x)| \leqslant \exp\{-2\gamma\cos^2\frac{\pi q}{2}|x|^q + o(|x|^q)\}$ as $x \to \pm\infty$, $x \in \mathbb{R}$.

In case (B), we want the absolute value of the function $F(z)$ to be dominated by $\exp\{-\beta\cos((\frac{\pi}{q}-\frac{\pi}{2})+q\theta)R^q + o(R^q)\}$ with some $\beta > 0$, on the rays $Re^{\pm i\theta}$. Since the ratio $\frac{-\cos(\frac{\pi}{q}-\frac{\pi}{2}+q\theta)}{(\sin\theta)^q}$ attains its maximum at the point $\theta = \frac{\pi}{q}$, we have

$$|F(z)| \leqslant \exp\Big\{\beta\frac{\cos(\frac{\pi}{q}-\frac{\pi}{2})}{(\sin\frac{\pi}{q})^q}|\operatorname{Im} z|^q + o(|\operatorname{Im} z|^q)\Big\},$$

while $|F(x)| \leqslant \exp\{-\beta\cos(\frac{\pi}{q}-\frac{\pi}{2})x^q + o(x^q)\}$ as $x \to +\infty$.

We define F by $F(z) = P(z)P(-\alpha^{1/q}z)$ with a suitable $\alpha \in (0,+\infty)$. Then

$$\log|F(Re^{\pm i\theta})| \leqslant -\gamma(\cos q\theta + \alpha\cos q(\pi-\theta))R^q + o(R^q)$$
$$= -\beta(\alpha)\cos(q\theta+\varphi(\alpha))R^q + o(R^q).$$

The phase shift $\varphi(\alpha)$ runs over the interval $(0, 2\pi-\pi q)$ as α runs from 0 to $+\infty$. But $0 < \frac{\pi}{q}-\frac{\pi}{2} < 2\pi-\pi q$ for $q \in (1,2)$, so the desired value of α really exists.

The condition $F\big|_{\mathbb{R}} \in S$ follows in both cases from the fast decrease of F in the strip $-1 < \operatorname{Im} z < 1$ (in case (B) it should be noted that $\cos(\frac{\pi}{q} - \frac{\pi}{2} + \pi q) > 0$).

The proof of the Beurling theorem stated at the beginning of this chapter is similar to the above reasoning and is also based on the analytic continuation technique, so we omit it. A detailed proof can be found in [8].

Comparing the statements of the Morgan theorem and Theorem 2.3, it is quite natural to ask the following question: What happens if the condition of fast distribution decrease is one-sided (i.e., it holds only on one of the semiaxis)? All I know about that is presented in §4 and §5. §4 is devoted to the case where f rapidly decreases on the whole real line $\mathbb{R}$, while $\widehat{f}$ only on $\mathbb{R}_-$. In this case the following uniqueness theorem, rather close to Theorem 2.3, can be proved.

Theorem 2.4. *Let $f \in L^2(\mathbb{R})$. If Φ majorizes the distribution of $|f|$ on the whole real line $\mathbb{R}$ and Ψ majorizes the distribution of $|\widehat{f}|$ on the left semiaxis $\mathbb{R}_-$, then the condition*

$$\varliminf_{a,b\to+\infty} e^{97ab\log ab}\left(\int_a^{+\infty}\Phi^2 + \int_b^{+\infty}\Psi^2\right) = 0$$

implies $f \equiv 0$.

The constant 97 in the exponent can be replaced by $1+\varepsilon$, but this is of no value because even the logarithmic factor might have been superfluous.

Nevertheless, Theorem 2.4 allows us to catch the best possible rate of decrease for majorants but on a rougher level (with order in place of type): this theorem implies that the estimates $\Phi(x) = O(e^{-\delta x^p})$, $\Psi(x) = O(e^{-\delta x^q})$ ($\delta > 0$, $\frac{1}{p} + \frac{1}{q} < 1$) are incompatible unless $f \equiv 0$.

If the condition of fast decrease is one-sided for both f and $\widehat{f}$, we know much less: in §5 it will be shown only that the intersections of $\mathbb{R}_-$ with the supports of f and $\widehat{f}$ cannot have finite measure simultaneously.

§2.4. The Turan lemma for $H^2(\mathbb{D})$ and a semi-one-sided uniqueness theorem

We shall start with proving the following lemma, which is also interesting in itself.

Lemma 2.5. *Let $f(z) = c_0 z^{m_0} + c_1 z^{m_1} + \dots + c_n z^{m_n} + \dots \in H^2(\mathbb{D})$ $(0 \leqslant m_0 < m_1 < \dots)$, $\|f\|_{H^2(\mathbb{D})} \leqslant 1$. Let $G(f) \overset{\text{def}}{=} \exp\int_{\mathbb{T}}\log|f|\,d\mu$ be the geometric mean of $|f|$ over the unit circumference $\mathbb{T}$. If $G(f) \leqslant G \leqslant 1$, then $|c_n| \leqslant e^n(1 + \log\frac{1}{G})^n G$.*

Proof. We prove the statement by induction on n. Without loss of generality, we can assume $m_0 = 0$ (otherwise we consider $z^{-m_0}f(z)$ instead of $f(z)$). Thus, the base of induction is reduced to the inequality $|f(0)| \leqslant G(f) \leqslant G$, which is immediate because $\log|f|$ is subharmonic. Now we assume that the estimate for the coefficient $|c_{n-1}|$ has been proved.

How can the estimate for $|c_n|$ be derived from this? The most natural way is to try to construct a function $g(z)$ with the following properties:

1) $\|g\|_{H^2(\mathbb{D})} \leqslant \|f\|_{H^2(\mathbb{D})}$;

2) the $(n-1)$-th nonvanishing coefficient of $g(z)$ is not too far from c_n;
3) $G(g)$ is only slightly greater than $G(f)$.

It turns out that one can put $g = (1-r^2)(f')_r$ with a suitable $r \in (0,1)$ (for a function h analytic in $\mathbb{D}$, we denote $h'(z) \overset{\text{def}}{=} \frac{d}{dz}h(z)$, $h_r(z) \overset{\text{def}}{=} h(rz)$, $z \in \mathbb{D}$).

In formal computations, it is convenient to use the mapping $\Phi_r \colon f \to (1-r^2)(f')_r$ defined on the set of all analytic functions in $\mathbb{D}$. If f admits a reasonable extension to $\mathbb{T}$, the Cauchy integral formula yields

$$\Phi_r f(z) = \int_{\mathbb{T}} \frac{(1-f^2)\zeta}{(\zeta - rz)^2} f(\zeta)\, d\mu(\zeta) = 2\int_{\mathbb{T}} \frac{(1-f^2)\zeta}{(\zeta - rz)^2} \operatorname{Re} f(\zeta)\, d\mu(\zeta)$$

for $z \in \overline{\mathbb{D}}$.

Thus, Φ_r is a linear integral operator with the kernel $k(z,\zeta) = \frac{(1-r^2)\zeta}{(\zeta-rz)^2}$. Besides, the function $|k(z,\zeta)| = \frac{(1-r^2)\zeta}{|\zeta-rz|^2}$ $(z,\zeta \in \mathbb{T})$ is precisely the Poisson kernel. Therefore, $\|\Phi_r\|_{L^p(\mathbb{T})\to L^p(\mathbb{T})} \leqslant 1$, $p \in [1,+\infty]$, and, in particular, $\|\Phi_r f\|_{H^2} \leqslant \|f\|_{H^2}$. Similarly, $\|\Phi_r f\|_{H^p} \leqslant 2\|\operatorname{Re} f\|_{L^p(\mathbb{T})}$, $p \in [1,+\infty]$. Let us look at the action of the operator Φ_r on the coefficients of an analytic function $f(z)$. If $f(z) = a_0 + a_1 z + \dots + a_k z^k + \dots$, then $(\Phi_r f)(z) = (1-r^2)a_1 + 2(1-r^2)ra_2 z + \dots + (k(1-r^2)r^{k-1}a_k)z^{k-1} + \dots$. Let $r^2 = \frac{k-1}{k+1}$, then the factor at the coefficient a_k is equal to $k(1-r^2)r^{k-1} = \frac{2k}{k+1}\left(\frac{k-1}{k+1}\right)^{\frac{k-1}{2}} \geqslant \frac{2}{e}$. Thus, if r is chosen suitably, then the absolute value of the $(n-1)$-th nonvanishing coefficient of the function $g = \Phi_r f$ is at most $\frac{2}{e}|c_n|$.

Finally, we shall estimate $G(\Phi_r f)$ provided that $f \in H^2$, $\|f\|_{H^2} \leqslant 1$, and $G(f) \leqslant G \leqslant 1$. We make use of a suitable factorization of $f(z)$. Namely, let $e^{\varphi(z)}$ be an outer function with modulus not exceeding $|f(z)|$ on $\mathbb{T}$ and such that the geometric mean of $e^{\varphi(z)}$ over $\mathbb{T}$ equals g and $|e^{\varphi(z)}| = |f(z)|$ whenever $|f(z)| \geqslant 1$ $(z \in \mathbb{T})$. For example, we can take $\varphi(z) = \varphi_1(z) + i\widetilde{\varphi}_1(z)$, where $\varphi_1(z)$ is the harmonic extension of $\log\max(|f(z)|,\beta)$ into the unit disk and $\widetilde{\varphi}_1(z)$ is its harmonic conjugate. If $G(f) \leqslant G \leqslant 1$, then the number $\beta \in [0,1]$ can actually be chosen so that $G(e^{\varphi}) = \exp\int_{\mathbb{T}} \varphi_1\, d\mu = G$. The function $f(z)$ is representable as a product $f(z) = e^{\varphi(z)}h(z)$ with $h(z) \in H^\infty(\mathbb{D})$, $\|h\|_\infty \leqslant 1$. Since

$$\begin{aligned}\Phi_r f &= (1-r^2)(f')_r = (1-r^2)((h')_r + (\varphi')_r h_r)e^{\varphi_r}\\ &= (\Phi_r h + (\Phi_r\varphi)h_r)e^{\varphi_r} \overset{\text{def}}{=} \psi e^{\varphi_r},\end{aligned}$$

we have $G(\Phi_r f) = G(\psi)G(e^{\varphi_r})$. But $G(e^{\varphi_r}) = \exp\int_{\mathbb{T}} \varphi_1(rz)\, d\mu(z) = \exp\int_{\mathbb{T}} \varphi_1\, d\mu = G(e^{\varphi}) = G$, and $G(\psi) \leqslant \|\psi\|_{L^1(\mathbb{T})} \leqslant 1 + \|\Phi_r\varphi\|_{L^1(\mathbb{T})} \leqslant 1 + 2\|\operatorname{Re}\varphi\|_{L^1(\mathbb{T})} = 1 + 2\int_{\mathbb{T}} |\varphi_1|\, d\mu$. Moreover, $\int_{\mathbb{T}} \varphi_1^+\, d\mu = \int_{\mathbb{T}} \log^+|f|\, d\mu \leqslant \int_{\mathbb{T}} \frac{|f|^2}{2e}\, d\mu \leqslant \frac{1}{2e}$, and $\int_{\mathbb{T}} \varphi_1\, d\mu = \log G$. Therefore, $\int_{\mathbb{T}} |\varphi_1|\, d\mu = -\int_{\mathbb{T}} \varphi_1\, d\mu + 2\int_{\mathbb{T}} \varphi_1^+\, d\mu \leqslant \log\frac{1}{G} + \frac{1}{e}$, and we obtain the inequality $G(\Phi_r f) < 2(1+\log\frac{1}{G})G$.

Besides, we have the trivial estimate $G(\Phi_r f) \leqslant \|\Phi_r f\|_{H^2} \leqslant \|f\|_{H^2} \leqslant 1$, whence $G(\Phi_r f) \leqslant \min(1, 2(1+\log\frac{1}{G})G) \overset{\text{def}}{=} G'$. Applying the induction assumption to the function g constructed above, we obtain $\frac{2}{e}|c_n| \leqslant e^{n-1}(1+\log\frac{1}{G'})^{n-1}G'$; since $G \leqslant G' \leqslant 2(1+\log\frac{1}{G})G$, we have $|c_n| \leqslant e^n(1+\log\frac{1}{G})^n G$, which proves the lemma.

Lemma 2.5 implies the following simple corollary.

Corollary. *If* $G \leqslant \frac{1}{e}$, then

$$|c_n| \leqslant (2e)^n \left(\log \frac{1}{G}\right)^n G \leqslant (4e)^n n! \frac{1}{n!} \left(\frac{1}{2} \log \frac{1}{G}\right)^n G \leqslant (4en)^n \sqrt{G}.$$

This estimate is obviously valid also if $G \in [\frac{1}{e}, +\infty]$. Compared with Lemma 2.5, the only advantage of this estimate is that now the right-hand side is monotone increasing as a function of G.

Later we shall encounter a function f which satisfies the restrictions $\|f\|_{H^2} \leqslant 1$, $\int_F |f|^2 \leqslant \rho^2 < 1$, $\mu(F) = \nu \geqslant \frac{1}{2}$. In this case the Jensen inequality implies

$$\begin{aligned} \log G(f) &= \int_F \log|f| + \int_{\mathbb{T} \neq F} \log|f| \leqslant \left(\log x \leqslant \frac{x^2}{2}\right) \\ &\leqslant \frac{1}{2}\int_{\mathbb{T}} |f|^2 + \frac{\nu}{2} \log\left(\frac{1}{\nu}\int_F |f|^2\right) \leqslant \frac{1}{2} + \frac{\nu}{2}\log\left(\frac{1}{\nu}\rho^2\right), \end{aligned}$$

and, therefore, $G(f) \leqslant e^{1/2}(\frac{1}{\nu})^{\nu/2}\rho^\nu \leqslant e\rho^\nu \leqslant e\rho^{1/2}$, which gives $|c_n| \leqslant (4en)^n e\rho^{1/4}$.

Now we derive a semi-one-sided uniqueness theorem from Lemma 2.5.

Let $f \in L^2(\mathbb{R})$, $\|f\|_{L^2(\mathbb{R})} = 1$. We assume that Φ majorizes the distribution of $|f|$ on the whole real line $\mathbb{R}$ and Ψ majorizes the distribution of $|\widehat{f}|$ on the semiaxis $\mathbb{R}_-$. Having fixed two positive numbers $a, b > 0$, we choose some sets $E \subset \mathbb{R}$ and $\Sigma \subset \mathbb{R}_-$ so that

$$\mu(E) = a, \qquad \mu(\Sigma) = b,$$
$$\int_{\mathbb{R} \setminus E} |f|^2 \leqslant \int_a^{+\infty} \Phi^2 \overset{\text{def}}{=} \Phi_a^2, \qquad \int_{\mathbb{R} \setminus \Sigma} |\widehat{f}|^2 \leqslant \int_b^{+\infty} \Psi^2 \overset{\text{def}}{=} \Psi_b^2.$$

We put $f_E = f\chi_E, \varepsilon = \frac{1}{4a}$ and introduce the random periodization g of the function f_E. As in §2, we have $\mu\{\, t \in (0,1) : g(t) = 0 \,\} \geqslant \frac{1}{2}$. Besides,

$$\int_{\mathbb{R}_- \setminus \Sigma} |\widehat{f}_E|^2 \leqslant 2\left(\int_{\mathbb{R}_- \setminus \Sigma} |\widehat{f}|^2 + \|f - f_E\|^2_{L^2(\mathbb{R})}\right) \leqslant 2(\Phi_a^2 + \Psi_b^2).$$

To consider the Fourier coefficients $\hat{g}_m$ with negative indices, we split these indices, as before, into two sets: $\mathfrak{M}_1 \overset{\text{def}}{=} \{\, m \in \mathbb{Z} : m < 0,\ m\varepsilon v \in \Sigma \,\}$ and $\mathfrak{M}_2 \overset{\text{def}}{=} \mathbb{Z}_- \setminus \mathfrak{M}_1$. Properties (B)–(D) of random periodization imply that $\mathbf{E}\operatorname{card}\mathfrak{M}_1 \leqslant \frac{b}{\varepsilon} = 4ab$, and, therefore, $\mathbf{P}\{\operatorname{card}\mathfrak{M}_1 \geqslant 12ab\} < \frac{1}{3}$. Besides,

$$\mathbf{E} \sum_{m \in \mathfrak{M}_2} |\hat{g}_m|^2 \leqslant 2\int_{\mathbb{R}_- \setminus \Sigma} |\widehat{f}_E|^2 \leqslant 4(\Phi_a^2 + \Psi_b^2),$$

whence

$$\mathbf{P}\left\{\sum_{m \in \mathfrak{M}_2} |\hat{g}_m|^2 > 12(\Phi_a^2 + \Psi_b^2)\right\} < \frac{1}{3}.$$

Finally,

$$\mathbf{E}\|g\|^2_{L^2(0,1)} \leqslant 2(\varepsilon a+1)\|f_E\|^2_{L^2(\mathbb{R})} \leqslant 3 \implies \mathbf{P}\{\|g\|_{L^2(0,1)} > 3\} < \frac{1}{3}.$$

Thus, with a positive probability the following 4 events take place simultaneously:

(1) $\operatorname{card}\mathfrak{M}_1 \leqslant 12ab$;

(2) $\sum_{m\in\mathfrak{M}_2} |\hat{g}_m|^2 \leqslant 12(\Phi_a^2+\Psi_b^2)$;

(3) $\|g\|_{L^2(0,1)} \leqslant 3$;

(4) $\mu\{\, t\in(0,1) : g(t)=0\,\} \overset{\text{def}}{=} \mu(F) \geqslant \frac{1}{2}$.

Putting $p(t)=\sum_{\mathbb{Z}\setminus\mathfrak{M}_2} \hat{g}_m e^{2\pi imt}$, we see that the function $p(z)$ has at most $12ab$ non-vanishing Fourier coefficients with negative indices, if any. Besides,

$$\int_F |p|^2\,d\mu = \int_F |g-p|^2\,d\mu \leqslant \int_{\mathbb{T}} |g-p|^2\,d\mu = \sum_{m\in\mathfrak{M}_2} |\hat{g}_m|^2 < 12(\Phi_a^2+\Psi_b^2),$$

and $\|p\|_{L^2(0,1)} \leqslant \|g\|_{L^2(0,1)} \leqslant 3$. Applying the estimate from the final remark after Lemma 2.4 with $\rho^2=\frac{3}{4}(\Phi_a^2+\Psi_b^2)$ to the function $\frac{1}{3}p$ (this is not perfectly well because there we dealt with functions from $H^2(\mathbb{D})$, but the reduction is trivial), we conclude that

$$\frac{1}{\sqrt{4a}}|\widehat{f}_E(0)| \leqslant |\hat{p}_0| = |\hat{g}_0| \leqslant 3e(4e\times 12ab)^{12ab}\left\{\frac{3}{4}(\Phi_a^2+\Psi_b^2)\right\}^{1/8},$$

whence $|\widehat{f}_E(0)| \leqslant 20\sqrt{a}(144ab)^{12ab}(\Phi_a^2+\Psi_b^2)^{1/8} \overset{\text{def}}{=} Q_0(a,b)$. Let $\lambda>0$. The same reasoning applied to the function $f(t)e^{2\pi i\lambda t}$ (with the same set E, and the set Σ shifted to the right by λ) leads to the inequality $|\widehat{f}_E(-\lambda)| \leqslant Q_0(a,b)$.

Now the absolute value of the Fourier image of the function $f(t)e^{-i\lambda t}$ has the distribution which cannot be majorized by Ψ on the left semi-axis because the interval $[0,\lambda]$ carrying an unknown distribution has been shifted there. But, surely, the distribution is majorized by the function

$$\Psi_{(\lambda)}(x) \overset{\text{def}}{=} \begin{cases} +\infty, & 0\leqslant x\leqslant\lambda,\\ \Psi(x-\lambda), & x>\lambda,\end{cases}$$

consequently,

$$|\widehat{f}_E(\lambda)| \leqslant 20\sqrt{a}(144a(b+\lambda))^{12a(b+\lambda)}(\Phi_a^2+\Psi_b^2)^{1/8} \overset{\text{def}}{=} Q_\lambda(a,b).$$

(We have replaced b by $b+\lambda$ and used the obvious identity $(\Psi_{(\lambda)})^2_{b+\lambda}=\Psi_b^2$).

Assuming that $\Phi_a^2+\Psi_b^2 < e^{-97ab\log ab}$ as $a,b\to+\infty$ along certain sets, we have $Q_\lambda(a,b)\to 0$ on the same set of pairs (a,b) for every $\lambda\geqslant 0$. Then $\widehat{f}_E(y)\to 0$ for every $y\in\mathbb{R}$, while $\widehat{f}_E\to\widehat{f}$ in the sense of $L^2(\mathbb{R})$. Thus, $\widehat{f}\equiv 0$ and $f\equiv 0$. The theorem is proved.

§2.5. The one-sided Amrein–Berthier theorem

We start with a simple approximation lemma.

Lemma 2.6. *Let $I \subset \mathbb{R}$ be an interval, $F \in L^2(I)$,* and let $e \subset I$ be a measurable subset of I. Suppose that for each $\rho \in (0, \frac{1}{2})$ we can find a function $\Phi_\rho \in H^2(\mathbb{C}_+)$ such that

1) $\int_{I \setminus e} |\Phi_\rho - F|^2 \, d\mu \leqslant C\rho^2$;
2) $\|\Phi_\rho\|_{H^2(\mathbb{C}_+)} \leqslant C\rho^{-1/6}$.

Then there exists a connected region $\Omega \subset \mathbb{C}_+$ with Lipschitz boundary and a function $\Phi \in H^2(\Omega)$ such that

1) $\mu(I \setminus \partial\Omega) \leqslant A\mu(e)$ (A is an absolute constant);
2) $\Phi = F$ a.e. on $\partial\Omega \cap I$.

Proof. Let $\Gamma(x)$ denote the set $\{\, z \in \mathbb{C}_+ : \operatorname{Im} z \geqslant |\operatorname{Re} z - x| \,\}$, i.e., the right angle with vertex $x \in \mathbb{R}$ bisected by a line parallel to the imaginary axis. Let u be the Poisson integral of the function χ_e. In other words, $u(z)$ is the harmonic measure of the set e with respect to the region $\mathbb{C}_+$ and the point $z \in \mathbb{C}_+$, i.e., $u(z) = \omega_{\mathbb{C}_+}(e, z)$. By the well-known weak type estimate of the nontangential maximal function, the measure of the open set $E = \{\, x \in \mathbb{R} : \sup_{\Gamma(x)} u > \frac{1}{4} \,\}$ does not exceed $A\mu(e)$. We put $\Omega \stackrel{\text{def}}{=} D_+ \cap \left(\bigcup_{x \in I \setminus E} \Gamma(x)\right)$, where D_+ is the open half-disk in $\mathbb{C}_+$ whose diameter is the interval I. The set Ω is the half-disk D_+ from which we cut off all isosceles right triangles based on the constituent intervals of E. Having drawn the corresponding picture, one can easily verify that Ω is a connected and simply connected region with Lipschitz boundary, and $\partial\Omega \cap I = I \setminus E$. For $z \in \Omega$ we have

$$\omega_{\mathbb{C}_+}(I \setminus e, z) = \omega_{\mathbb{C}_+}(I, z) - \omega_{\mathbb{C}_+}(e, z) \geqslant \frac{1}{2} - u(z) \geqslant \frac{1}{4}.$$

Now we estimate the norm $\|\Phi_\rho - \Phi_{\rho/2}\|_{H^2(\Omega)}$. (As usual, we define the $H^2(\Omega)$-norm of a function $\varphi \in H^2(\Omega)$ by $\|\varphi\|_{H^2(\Omega)} \stackrel{\text{def}}{=} \inf\{\, u(\zeta_0) : \Delta u = 0 \text{ in } \Omega,\ |\varphi|^2 \leqslant u \text{ in } \Omega \,\}$, where ζ_0 is a fixed point from Ω. Another form of this definition is $\|\varphi\|_{H^2(\Omega)} \stackrel{\text{def}}{=} \|\varphi \circ \psi\|_{H^2(\mathbb{D})}$, where ψ is a conformal mapping of the unit disk $\mathbb{D}$ onto Ω with $\psi(0) = \zeta_0$.) It is well known that $\|\varphi\|^2_{H^2(\Omega)} = \lim_{n\to\infty} \int_{\partial\Omega_n} |\varphi(\zeta)|^2 \, d\omega_{\Omega_n}(\zeta, \zeta_0)$, where $\Omega_n \uparrow \Omega$ are simply connected regions with smooth boundaries containing ζ_0 and relatively compact in Ω. The function $\varphi = \Phi_\rho - \Phi_{\rho/2}$ belongs to $H^2(\mathbb{C}_+)$, and $\|\varphi\|_{H^2(\mathbb{C}_+)} \leqslant \|\Phi_\rho\|_{H^2(\mathbb{C}_+)} + \|\Phi_{\rho/2}\|_{H^2(\mathbb{C}_+)} \leqslant 3C\rho^{1/6}$. Since the function $\log|\varphi|$ is subharmonic in $\mathbb{C}_+$, it admits the following estimate at the points $\zeta \in \partial\Omega_n$:

$$\begin{aligned}
\log|\varphi(\zeta)| &\leqslant \int_{\mathbb{R}} \log|\varphi(x)| P(\zeta, x) \, dx \\
&= \int_{e'} \log|\varphi(x)| P(\zeta, x) \, dx + \int_{\mathbb{R} \setminus e'} \log|\varphi(x)| P(\zeta, x) \, dx \\
&\leqslant \gamma(\zeta) \log\left(\int_{e'} P(\zeta, x)|\varphi(x)|^2 \, dx\right)^{\frac{1}{2}} \\
&\quad + (1 - \gamma(\zeta)) \log\left(\int_{\mathbb{R} \setminus e'} P(\zeta, x)|\varphi(x)|^2 \, dx\right)^{\frac{1}{2}}.
\end{aligned}$$

(Here P is the Poisson kernel for $\mathbb{C}_+$, $e' \stackrel{\text{def}}{=} I \setminus e$, and $\gamma(\zeta) \stackrel{\text{def}}{=} \omega_{\mathbb{C}_+}(I \setminus e, \zeta)$). Replacing in the last sum $\int_{\mathbb{R}\setminus e'}$ by $\int_{\mathbb{R}}$, exponentiating both sides, and taking into account that $\gamma(\zeta) \geqslant \frac{1}{4}$, we obtain

$$|\varphi(\zeta)|^2 \leqslant \left(\int_{e'} P(\zeta, x)|\varphi(x)|^2\,dx\right)^{\frac{1}{4}} \left(\int_{\mathbb{R}} P(\zeta, x)|\varphi(x)|^2\,dx\right)^{\frac{3}{4}}.$$

Therefore,

$$\begin{aligned}
&\int_{\partial\Omega_n} |\varphi(\zeta)|^2\,d\omega_{\Omega_n}(\zeta, \zeta_0)\\
&\qquad\leqslant \int_{\partial\Omega_n} \left(\int_{e'} P(\zeta, x)|\varphi(x)|^2\,dx\right)^{\frac{1}{4}} \left(\int_{\mathbb{R}} P(\zeta, x)|\varphi(x)|^2\,dx\right)^{\frac{3}{4}} d\omega_{\Omega_n}(\zeta, \zeta_0)\\
&\qquad\leqslant \left(\int_{\partial\Omega_n} \left(\int_{e'} P(\zeta, x)|\varphi(x)|^2\,dx\right) d\omega_{\Omega_n}(\zeta, \zeta_0)\right)^{\frac{1}{4}}\\
&\qquad\quad\times \left(\int_{\partial\Omega_n} \left(\int_{\mathbb{R}} P(\zeta, x)|\varphi(x)|^2\,dx\right) d\omega_{\Omega_n}(\zeta, \zeta_0)\right)^{\frac{3}{4}}\\
&\qquad= \left(\int_{e'} P(\zeta_0, x)|\varphi(x)|^2\,dx\right)^{\frac{1}{4}} \left(\int_{\mathbb{R}} P(\zeta_0, x)|\varphi(x)|^2\,dx\right)^{\frac{3}{4}}
\end{aligned}$$

since $P(\zeta, x)$ is harmonic with respect to $\zeta \in \Omega_n$. Consequently,

$$\begin{aligned}
\|\varphi\|^2_{H^2(\Omega)} &\leqslant \left(\int_{e'} P(\zeta_0, x)|\varphi(x)|^2\,dx\right)^{\frac{1}{4}} \left(\int_{\mathbb{R}} P(\zeta_0, x)|\varphi(x)|^2\,dx\right)^{\frac{3}{4}}\\
&\leqslant \max_{x\in\mathbb{R}} P(\zeta_0, x) \times \left(\int_{e'} |\varphi(x)|^2\,dx\right)^{\frac{1}{4}} \left(\int_{\mathbb{R}} |\varphi(x)|^2\,dx\right)^{\frac{3}{4}}\\
&\leqslant C'\rho^{\frac{1}{2}}\rho^{-\frac{1}{3}} = C'\rho^{\frac{1}{6}},
\end{aligned}$$

i.e., $\|\Phi_\rho - \Phi_{\rho/2}\|_{H^2(\Omega)} \leqslant C'\rho^{\frac{1}{12}}$. This estimate immediately implies that the sequence $\{\Phi_{2^{-k}}\}_{k=1}^{\infty}$ is fundamental in $H^2(\Omega)$, whence there exists a limit $\lim_{H^2(\Omega)} \Phi_{2^{-k}} \stackrel{\text{def}}{=} \Phi$. Since every density point of e belongs to E, we have $\Phi_{2^{-k}} \to F$ on $\partial\Omega \cap I = I \setminus E$ a.e. with respect to the Lebesgue measure on $\mathbb{R}$ or, what is the same, with respect to the arc length on $\partial\Omega$. But $\partial\Omega$ is rectifiable, and, therefore, $\Phi_{2^{-k}} \to F$ a.e. with respect to the harmonic measure $\omega_\Omega(\cdot, \zeta_0)$. So, $\Phi = F$ a.e. on $\partial\Omega \cap I$ with respect to both of these measures (in the sense of nontangential boundary values). Thus, Lemma 2.6 is proved.

Now we take $f \in L^2(\mathbb{R})$ and suppose that the sets $\operatorname{spec}_- f \stackrel{\text{def}}{=} \operatorname{spec} f \cap \mathbb{R}_-$ and $\operatorname{supp}_- f \stackrel{\text{def}}{=} \operatorname{supp} f \cap \mathbb{R}_-$ have finite measure; we want to show that $f \equiv 0$. If $\mu(\operatorname{supp}_- f) = 0$, then $\widehat{f}$ is the boundary function of some function from $H^2(\mathbb{C}_-)$, and there is nothing to prove. Let $\mu(\operatorname{supp}_- f) > 0$. Replacing, if needed, $f(x)$ by $f(Bx)$, where $B > 0$ is large enough, we can assume that

a) $\mu(\operatorname{supp}_- f) \leqslant \frac{1}{2}$;

b) $\mu(\mathrm{supp}_- f \cap (-\infty, -1)) \leqslant \frac{\delta}{2}\mu(\mathrm{supp}_- f \cap (-1, 0))$, where δ is a small positive number to be chosen later. Replacing $f(x)$ by $e^{i\lambda x} f(x)$ with sufficiently large $\lambda > 0$, we can also assume that

c) $\mu(\mathrm{spec}_- f) \leqslant \frac{1}{7}$.

Let $\Sigma \stackrel{\mathrm{def}}{=} \mathrm{spec}_- f$. The set $\Sigma_1 \stackrel{\mathrm{def}}{=} \bigcup_{k\in\mathbb{Z}}(\Sigma + k)$ is a 1-periodic set containing Σ. We set $\sigma \stackrel{\mathrm{def}}{=} \mu(\Sigma_1 \cap (0, 1))$, then $\sigma \leqslant \mu(\mathrm{spec}_- f) \leqslant \frac{1}{7}$. Let $\rho \leqslant \frac{1}{2}$. There exists a trigonometric polynomial $q(\xi) = q_\rho(\xi) = \sum_{k=0}^{n} q_k e^{-2\pi i k\xi}$ with the following properties:

1) $q_0 = 1$;

2) $\max_{\xi\in\mathbb{R}} |q(\xi)| \leqslant \rho^{-1/6}$;

3) $\int_{\mathbb{R}_-} |\widehat{f}(\xi)|^2 |q(\xi)|^2\, d\xi \leqslant 2\|f\|_{L^2(\mathbb{R})}\rho^2$.

Indeed, consider an outer function $Q(z)$ such that

$$|Q(e^{2\pi i t})| = \begin{cases} \rho, & t \in \Sigma_1, \\ \rho^{-\frac{\sigma}{1-\sigma}}, & t \notin \Sigma_1. \end{cases}$$

¿From this definition it follows that $Q(0) = 1$, $\sup_{|z|\leqslant 1} |Q(z)| \leqslant \rho^{-1/6}$, and

$$\int_{\mathbb{R}_-} |\widehat{f}(\xi)|^2 |Q(e^{2\pi i\xi})|^2\, d\xi \leqslant \rho^2 \int_{\Sigma} |\widehat{f}(\xi)|^2\, d\xi \leqslant \|f\|_{L^2(\mathbb{R})}\rho^2.$$

Now it is clear that we can take $q(\xi) = \overline{Q_n(e^{2\pi i\xi})}$, where Q_n is the Fejer polynomial for the function Q of a sufficiently large degree n.

Let $\Phi_\rho \stackrel{\mathrm{def}}{=} (q\widehat{f}\xi_{\mathbb{R}_+})^\vee = (q\widehat{f})^\vee - (q\widehat{f}\chi_\Sigma)^\vee$. We have $(q\widehat{f})^\vee(x) = \sum_{k=0}^{n} \hat{q}_k f(x-k) = f(x)$ for every $x \in (-1, 0) \smallsetminus \bigcup_{k=1}^{\infty}(\mathrm{supp}_- f + k)$. If $e \stackrel{\mathrm{def}}{=} (-1, 0) \cap (\bigcup_{k=1}^{\infty}(\mathrm{supp}_- f + k))$, then, obviously, $\mu(e) \leqslant \mu(\mathrm{supp}_- f \cap (-\infty, -1))$. Moreover,

$$\|q\widehat{f}\chi_\Sigma\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}_-} |\widehat{f}(\xi)|^2 |q(\xi)|^2\, d\xi \leqslant 2\|f\|_{L^2(\mathbb{R})}\rho^2.$$

Applying Lemma 2.6 to $I = (-1, 0)$, $F = f$, and the set e defined above, we get a region Ω and a function $\Phi \in H^2(\Omega)$ such that $\Phi = f$, a.e. on $\partial\Omega \cap I$ and $\mu(I \smallsetminus \partial\Omega) \leqslant A\mu(e)$. Now from a) and b) it follows that if $\delta A < 1$,then $\partial\Omega$ intersects each of the sets $(-1, 0) \cap \mathrm{supp}\, f$ and $(-1, 0) \smallsetminus \mathrm{supp}\, f$ by a set of positive measure, and, therefore, Φ vanishes on a subset of positive length in $\partial\Omega$, not being zero identically. This contradiction completes the proof.

Chapter 3. Functions with nearly linear dependent small shifts and the Zygmund theorem

Our main purpose in this chapter is to prove the quantitative version of the Zygmund theorem. However, we start with a study of some auxiliary class of functions that behave "in the small" like exponential polynomials with pure imaginary exponents.

Definition. Let $v \in \mathbb{N}$, $\tau, \varkappa > 0$. We shall say that $f \in EP^n_{\mathrm{loc}}(\tau, \varkappa)$ if $f \in L^2(\mathbb{R})$ and for every $t \in (0, \tau)$ the functions $f_{kt}(\cdot) \overset{\mathrm{def}}{=} f(\cdot + kt)$ $(k = 0, \dots, n)$ are nearly linear dependent, which means, more precisely, that there exist numbers $a_0(t), \dots, a_n(t) \in \mathbb{C}$ such that $\sum_{k=0}^n |a_k(t)|^2 = 1$ and $\left\| \sum_{k=0}^n a_k(t) f_{kt} \right\|_{L^2(\mathbb{R})} \leqslant \varkappa$. Since any $n+1$ shifts on an exponential polynomial of order n are linear dependent, the above property really makes the function f somewhat similar to an exponential polynomial. This relationship is rather close in the following sense:

1) Every exponential polynomial p of order n with spectrum $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ satisfies the equation $\hat{p}(\cdot) \prod_{j=1}^n (\cdot - \lambda_j) = 0$; similarly, every function $f \in EP^n_{\mathrm{loc}}(\tau, \varkappa)$ satisfies the inequality

$$\left\| \widehat{f}(\cdot) \prod_{j=1}^n \theta_\tau(\cdot - \lambda_j) \right\|_{L^2(\mathbb{R})} \leqslant \{20(n+1)\}^{2n} \varkappa$$

with suitable $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ (here $\theta_\tau(x) \overset{\mathrm{def}}{=} \min(1, \tau|x|)$). It is quite natural to call the set of these numbers $\lambda_1, \dots, \lambda_n$ the essential spectrum of the function f (the latter is not uniquely determined, of course, but the arbitrariness in its choice is very like that in the choice of n points of spectrum for an exponential polynomial whose true order is strictly less than n).

The proof of the above statement is given in §3.1.

2) On intervals of length comparable with τ a function $f \in EP^n_{\mathrm{loc}}(\tau, \varkappa)$ behaves like an exponential polynomial of order n. More precisely, in §3.2 we show that for each essential spectrum $\lambda_1, \dots, \lambda_n$ of the function f there exists an "error function" $\Phi \in L^2(\mathbb{R})$ with the properties that its L^2-norm does not exceed $\{200(n+1)\}^{2n}\varkappa$ and for every interval $J \subset \mathbb{R}$ of length $|J| \leqslant M\tau$ $(M > 1)$ one can find an exponential polynomial $p^{(J)}$ with spectrum $\lambda_1, \dots, \lambda_n$ satisfying the inequality $|f(x) - p^{(J)}(x)| \leqslant M^n \Phi(x)$ for all $x \in J$.

If we think n to be fixed and let the number $\varkappa$ tend to zero, the described properties are characteristic. Indeed, if we define $a_k(t)$ by $\prod_{j=1}^n \left(z - e^{2\pi i \lambda_j}\right) = \sum_{k=0}^n a_k(t) z^k$, then the inequality $\left\| \widehat{f}(\cdot) \prod_{j=1}^n \theta_\tau(\cdot - \lambda_j)) \right\|_{L^2(\mathbb{R})} \leqslant \varkappa$ implies

$$\begin{aligned} \left\| \sum_{k=0}^n a_k(t) f_{kt} \right\|_{L^2(\mathbb{R})} &= \left\| \widehat{f}(\cdot) \prod_{j=1}^n \left(e^{2\pi i t \cdot} - e^{2\pi i t \lambda_j}\right) \right\|_{L^2(\mathbb{R})} \\ &\leqslant (2\pi)^n \left\| \widehat{f}(\cdot) \prod_{j=1}^n \theta_\tau(\cdot - \lambda_j) \right\|_{L^2(\mathbb{R})} \leqslant (2\pi)^n \varkappa \end{aligned}$$

since $\left| e^{2\pi i t \lambda} - e^{2\pi i t \lambda_j} \right| \leqslant 2\pi \theta_\tau(\lambda - \lambda_j)$ for every $t \in (0, \tau)$, $\lambda \in \mathbb{R}$. But, if $a_k(t)$ are those chosen above, then it is evident that $\sum_{k=0}^n |a_k(t)|^2 \geqslant |a_n(t)|^2 = 1$, so $f \in EP^n_{\mathrm{loc}}\left(\tau, (2\pi)^n \varkappa\right)$.

On the other hand, if a function f can be approximated on every interval of length $n\tau$ by an exponential polynomial with spectrum $\lambda_1, \dots, \lambda_n$, and if the error of the approximation is majorized by a function Φ, $\|\Phi\|_{L^2(\mathbb{R})} \leqslant \varkappa$, then with the same choice of $a_k(t)$, for each

$x \in \mathbb{R}$ we have

$$\begin{aligned}\Big|\sum_{k=0}^{n} a_k(t) f_{kt}(x)\Big| &= \Big|\sum_{k=0}^{n} a_k(t) f(x+kt)\Big| = \Big|\sum_{k=0}^{n} a_k(t)\big(f(x+kt) - p(x+kt)\big)\Big| \\ &\leqslant \sum_{k=0}^{n} |a_k(t)|\Phi(x+kt) \leqslant \Big(\sum_{k=0}^{n} |a_k(t)|^2\Big)^{\frac12} \Big(\sum_{k=0}^{n} \Phi(x+kt)^2\Big)^{\frac12}.\end{aligned}$$

Here p stands for an exponential polynomial approximating f on the interval $[x, x+n\tau]$, and, clearly, the latter contains all the points of the form $x + kt$ ($k = 0, \dots, n, t \in (0, \tau)$). Squaring the two sides of the last inequality and integrating over $\mathbb{R}$, we obtain the estimate

$$\Big\|\sum_{k=0}^{n} a_k(t) f_{kt}\Big\|_{L^2(\mathbb{R})} \leqslant \sqrt{n+1}\Big(\sum_{k=0}^{n} |a_k(t)|^2\Big)^{\frac12} \|\Phi\|_{L^2(\mathbb{R})} \leqslant \sqrt{n+1}\Big(\sum_{k=0}^{n} |a_k(t)|^2\Big)^{\frac12} \varkappa,$$

which coincides (up to normalization) with the condition $f \in EP^n_{\mathrm{loc}}(\tau, \sqrt{n+1}\varkappa)$.

In §3.3 we try to find an analogue of the Turan lemma for functions of class $EP^n_{\mathrm{loc}}(\tau, \varkappa)$. It will be shown that, if $f \in EP^n_{\mathrm{loc}}(\tau, \varkappa)$ is small (say, in L^2-sense) outside a set E of finite measure, f cannot be too large on a certain part of E, the measure of this part depending on how well E can be approximated by a union of intervals of length comparable with τ, or, in other words, how much E changes under a shift by $n\tau$. In §3.4 and §3.5 we apply the technique developed in §§3.1–3.3 to prove quantitative versions of the Zygmund theorem for the unit circumference and the real line, respectively. It should be noted that in §§3.1–3.3 we deal with functions defined on the real line because this allows us to avoid digressions to unessential technical details and analysis of various cases. Therefore, §3.5 should, perhaps, precede §3.4. I placed the case of circumference first merely because theorems of this kind for Fourier series are more usual than those for Fourier integrals. The changes that should be made in the arguments of §§3.1–3.3 to deal with the case of circumference are pointed out in §3.4.

§3.1. A spectral description of the class $EP^n_{\mathrm{loc}}(\tau, \varkappa)$

The main result of this section has already been stated in the introduction to this chapter. It remains only to prove it. Our reasoning will be based on a simple observation: because of the identity

$$\Big\|\sum_{k=0}^{n} a_k(t) f_{kt}\Big\|_{L^2(\mathbb{R})} = \big\|\widehat{f}(\cdot) q_t(e^{2\pi i t \cdot})\big\|_{L^2(\mathbb{R})},$$

where $q_t(z) \overset{\mathrm{def}}{=} \sum_{k=0}^{n} a_k(t) z^k$, $z \in \mathbb{C}$, the norm of the linear combination $\sum_{k=0}^{n} a_k(t) f_{kt}$ can be small only if large values of $\widehat{f}$ are multiplied by small values of q_t.

First we show that $\widehat{f}$ is essentially concentrated on n intervals of length about $\frac{1}{n^2\tau}$. More precisely, the following statement is true.

Lemma 3.1. *Let* $\delta \leqslant \frac{1}{8n(n+1)}$. If $f \in EP^n_{\mathrm{loc}}(\tau, \varkappa)$, then there exist n intervals $I_1, \dots, I_n$, each of length $\frac{2\delta}{\tau}$, such that

$$\int_{\mathbb{R}\setminus\cup_1^n I_k} |\widehat{f}|^2 \leqslant 4(n+1)\left(\frac{4e}{\delta}\right)^{2n} \varkappa^2.$$

Proof. The shift operator being continuous in $L^2(\mathbb{R})$, we can assume the functions $a_k(t)$ in the definition of the class $EP^n_{\mathrm{loc}}(\tau, \varkappa)$ to be piecewise constant and, thus, measurable. Averaging the inequality $\int_{\mathbb{R}} |\widehat{f}(\lambda)|^2 |q_t(e^{2\pi it\lambda})|^2 \, d\lambda \leqslant \varkappa^2$ over the interval $(0, \tau)$, we obtain

$$\int_{\mathbb{R}} |\widehat{f}(\lambda)|^2 \left\{ \int_0^\tau |q_t(e^{2\pi it\lambda})|^2 \frac{dt}{\tau} \right\} d\lambda \stackrel{\mathrm{def}}{=} \int_{\mathbb{R}} |\widehat{f}(\lambda)|^2 \rho^2(\lambda)\, d\lambda \leqslant \varkappa^2.$$

Let us show that the function $\rho^2(\lambda)$ cannot be very small simultaneously at $n+1$ points $\xi_1 < \dots < \xi_{n+1}$ satisfying $\xi_{j+1} - \xi_j \geqslant \frac{2\delta}{\tau}$ for $j = 1, \dots, n$. Indeed, for every $t \in \left(\frac{\tau}{2}, \tau\right)$, the set $S_t \stackrel{\mathrm{def}}{=} \left\{ \xi \in \mathbb{R} : \left|q_t\left(e^{2\pi it\xi}\right)\right| < \left(\frac{\delta}{4e}\right)^n \right\}$ is a union of at most n $\frac{1}{t}$-periodic systems of intervals of the form $\left\{ J + \frac{k}{t} : k \in \mathbb{Z} \right\}$, where J is an interval of length $\frac{\delta}{t} < \frac{2\delta}{\tau}$ or less. To see this, one can argue as follows. Consider the set $U \stackrel{\mathrm{def}}{=} \left\{ z \in \mathbb{T} : |q_t(z)| < \left(\frac{\delta}{4e}\right)^n \right\}$. Its boundary is determined by the equation $|q_t(z)|^2 = \left(\frac{\delta}{4e}\right)^{2n}$ or, which is the same, by

$$\left(\sum_{k=0}^n a_k(t) z^k\right)\left(\sum_{k=0}^n \overline{a}_k(t) z^{-k}\right) = \left(\frac{\delta}{4e}\right)^{2n}$$

(we have used the identity $\overline{z} = 1/z$ for $z \in \mathbb{T}$). The left-hand side is a rational function of degree at most $2n$. Hence the number of solutions of this equation does not exceed $2n$, and, therefore, U consists of at most n arcs. The length of each of them can be estimated from above by $2\pi\delta$, in accordance with the Turan lemma (which is applied here in its simplest form: q_t is a polynomial of degree n) if we take into account that

$$\sup_{z\in\mathbb{T}} |q_t(z)| \geqslant \left(\int_{\mathbb{T}} |q_t|^2 \, d\mu\right)^{1/2} = \left(\sum_{k=0}^n |a_k(t)|^2\right)^{1/2} = 1.$$

On the real line each of these arcs becomes a system of intervals of the above kind.

If

$$\sum_{j=1}^{n+1} \left|q_t\left(e^{2\pi it\xi_j}\right)\right|^2 < \left(\frac{\delta}{4e}\right)^{2n}$$

for some $t \in (0, \tau)$, then all the points ξ_j belong to S_t. By virtue of the Dirichlet principle, two of them, say, $\xi_{j'}$ and $\xi_{j''}$ $(j' < j'')$, lie in the same periodic system of intervals constituting the set S_t. This means that there exists $k \in \mathbb{Z}$ such that $\left|\xi_{j''} - \xi_{j'} - \frac{k}{t}\right| \leqslant \frac{\delta}{t} < \frac{2\delta}{\tau}$. Since $\xi_{j''} - \xi_{j'} > \frac{2\delta}{\tau}$, we have $k > 0$. Let G be the union of $\frac{n(n+1)}{2}$ intervals of length $\frac{4\delta}{\tau}$ centered at the points $\xi_{j''}-\xi_{j'}$ $(1\leqslant j'<j''\leqslant n)$. The above condition is equivalent to the following one: there exists $k\in\mathbb{N}$ such that $k/t\in G$ or, which is the same, $\sum_{k=1}^{\infty}\chi_G\big(\frac{k}{t}\big)\geqslant 1$. The lattice averaging lemma (with $\varphi=\chi_G$ and $\varepsilon=\tau/2$) implies

$$\int_{\tau/2}^{\tau}\sum_{k>0}\chi_G\big(\frac{k}{t}\big)\,dt=\frac{\tau}{2}\int_1^2\sum_{k>0}\chi_G\left(\frac{k}{(\tau/2)v}\right)dv$$
$$\leqslant\frac{\tau}{2}\times 4\frac{\tau}{2}\int_{\mathbb{R}}\chi_G(\xi)\,d\xi\leqslant\frac{\tau}{2}\times 4\frac{\tau}{2}\times\frac{1}{4\tau}=\frac{\tau}{4},$$

since $\mu(G)\leqslant\frac{n(n+1)}{2}\frac{4\delta}{\tau}=\frac{2\delta n(n+1)}{\tau}\leqslant\frac{1}{4\tau}$ (we have assumed that $\delta\leqslant\frac{1}{8n(n+1)}$). Thus, the measure of the set of all $t\in(\tau/2,\tau)$ satisfying $\sum_{j=1}^{n+1}|q_t(e^{2\pi it\xi_j})|^2<\big(\frac{\delta}{4e}\big)^{2n}$ does not exceed $\tau/4$. Then we have

$$\sum_{j=1}^{n+1}\rho^2(\xi_j)=\int_0^{\tau}\sum_{j=1}^{n+1}\big|q_t\big(e^{2\pi it\xi_j}\big)\big|^2\frac{dt}{\tau}\geqslant\frac{1}{4}\left(\frac{\delta}{4e}\right)^{2n}.$$

Now let $S\overset{\text{def}}{=}\big\{\lambda\in\mathbb{R}:\rho^2(\lambda)<\frac{1}{4(n+1)}\big(\frac{\delta}{4e}\big)^{2n}\big\}$. It is obvious that

$$\int_{\mathbb{R}\setminus S}|\widehat{f}(\lambda)|^2\,d\lambda\leqslant 4(n+1)\left(\frac{4e}{\delta}\right)^{2n}\varkappa^2.$$

On the other hand, we cannot choose $n+1$ points $\xi_1<\cdots<\xi_{n+1}$ from S so that $\xi_{j+1}-\xi_j\geqslant\frac{2\delta}{\tau}$ for each $j=1,\dots,n$, which precisely means that S can be covered by n intervals of length $\frac{2\delta}{\tau}$. Lemma 3.1 is proved.

Let $\widetilde{S}=\cup_1^n\widetilde{I}_j$, where $\widetilde{I}_j$ is a $\frac{\delta}{\tau}$-neighborhood of the interval I_j, i.e., the interval of length $\frac{4\delta}{\tau}$ centered at the same point. By the same argument as above, it can easily be shown that there exists $t_0\in(\tau/2,\tau)$ such that any arithmetic progression with difference t_0 contains at most one point from $\widetilde{S}$. Indeed, the existence of an arithmetic progression with difference $1/t$ containing 2 or more points from $\widetilde{S}$ is equivalent to the existence of a number $k\in\mathbb{Z}\setminus\{0\}$ such that $k/t\in\widetilde{G}\overset{\text{def}}{=}\widetilde{S}-\widetilde{S}$. Taking into account that $\widetilde{G}$ is symmetric with respect to the origin, we obtain $\sum_{k\in\mathbb{Z}\setminus\{0\}}\chi_{\widetilde{G}}(k/t)\geqslant 2$ (together with k/t, the set $\widetilde{G}$ must contain $-k/t$). But $\mu(\widetilde{G})\leqslant\frac{8\delta}{\tau}n^2$, and now it follows from the lattice averaging lemma that

$$\int_{\tau/2}^{\tau}\sum_{k\in\mathbb{Z}\setminus\{0\}}\chi_{\widetilde{G}}(k/t)\,dt\leqslant\tau^2\mu(\widetilde{G})\leqslant 8\delta n^2\tau<\tau,$$

because $\delta\leqslant\frac{1}{8n(n+1)}<\frac{1}{8n^2}$. Thus, the measure of the set of "bad" $t\in(\tau/2,\tau)$ is strictly less than $\tau/2$, which proves the claim.

Now we show that for every $\lambda\in\cup_1^nI_j$ the absolute value of the factor $q_{t_0}(e^{2\pi it_0\lambda})$ can be estimated from below by $\prod_{j=1}^n\theta_\tau(\lambda-\lambda_j)$ if the numbers $\lambda_j\in\mathbb{R}$ are suitably chosen. Let $z_1,\dots,z_n$ be all complex zeros of the polynomial $q_{t_0}(z)$. We have $q_{t_0}(z)=\alpha\prod_{j=1}^n(z-z_j)$.

Let $\zeta_j \overset{\text{def}}{=} \frac{z_j}{|z_j|}$, $p(z) \overset{\text{def}}{=} \prod_{j=1}^n (z-\zeta_j)$. Since the ratio $\left|\frac{z-\zeta_j}{z-z_j}\right|$ attains its maximum $\frac{2}{1+|z_j|}$ at the point $z=-\zeta_j$, we have

$$\left|\frac{p(z)}{q_{t_0}(z)}\right| \leqslant \frac{1}{|\alpha|}\prod_{j=1}^n \frac{2}{1+|z_j|}$$

for every $z\in\mathbb{T}$. But, as shown above, $\sup_{\mathbb{T}}|q_{t_0}|\geqslant 1$. Therefore $|\alpha|\geqslant \prod_{j=1}^n \frac{1}{1+|z_j|}$, whence we obtain the inequality $|p(z)|\leqslant 2^n|q_{t_0}(z)|$ $(z\in\mathbb{T})$. Since $\|\widehat{f}(\cdot)q_{t_0}(e^{2\pi i t_0\cdot})\|_{L^2(\mathbb{R})}\leqslant\varkappa$, we have $\|\widehat{f}(\cdot)p(e^{2\pi i t_0\cdot})\|_{L^2(\mathbb{R})}\leqslant 2^n\varkappa$. The real zeros of the trigonometric polynomial $p(e^{2\pi i t_0\cdot})$ form n arithmetic progressions with difference $1/t_0$. We denote by $\lambda_1,\dots,\lambda_m$ the zeros lying in $\widetilde{S}$. It should be noticed that $m\leqslant n$ because each of the above progressions has at most one point in $\widetilde{S}$. Each λ_j $(j=1,\dots,m)$ corresponds to the root $e^{2\pi i t_0\lambda_j}$ of the polynomial $p(z)$. To the set $\lambda_1,\dots,\lambda_m$ we add certain numbers $\lambda_{m+1},\dots,\lambda_n$, so that the set $\{e^{2\pi i t_0\lambda_j}\}_{j=1}^n$ coincide with the set of all roots of p. Then

$$\left|p\big(e^{2\pi i t_0\lambda}\big)\right| = 2^n\prod_{j=1}^n\left|\sin\pi t_0(\lambda-\lambda_j)\right| \qquad (\lambda\in\mathbb{R}).$$

Since $|\sin\pi x|\geqslant 2|x|$ $(x\in\left[-\frac12,\frac12\right])$, the factor $|\sin\pi t_0(\lambda-\lambda_j)|$ must be at least $2t_0 d_j(\lambda)$, where $d_j(\lambda)$ is the distance from λ to the nearest point of the form λ_j+k/t_0, $k\in\mathbb{Z}$. But if $\lambda\in\cup_{k=1}^n I_k$, there can be at most one point of this kind whose distance from λ is less than δ/τ, namely, λ_j itself. Hence, for every $\lambda\in\cup_{k=1}^n I_k$ we get

$$d_j(\lambda)\geqslant\min\left(\frac{\delta}{\tau},|\lambda-\lambda_j|\right)\geqslant\frac{\delta}{\tau}\min(1,\tau|\lambda-\lambda_j|)=\frac{\delta}{\tau}\theta_\tau(\lambda-\lambda_j),$$

because $\delta\leqslant\frac{1}{8n(n+1)}<1$. The inequality $t_0\geqslant\frac{\tau}{2}$ implies $|\sin\pi t_0(\lambda-\lambda_j)|\geqslant 2t_0 q(\frac{\delta}{\tau})\theta_\tau(\lambda-\lambda_j)\geqslant\delta\theta_\tau(\lambda-\lambda_j)$. Therefore, $|p(e^{2\pi i t_0\lambda})|\geqslant 2^n\delta^n\prod_{j=1}^n\theta_\tau(\lambda-\lambda_j)$, and

$$\int_{\bigcup_1^n I_k}|\widehat{f}(\lambda)|^2\left|\prod_{j=1}^n\theta_\tau(\lambda-\lambda_j)\right|^2 d\lambda\leqslant 2^{-2n}\delta^{-2n}\|\widehat{f}(\cdot)p(e^{2\pi i t_0\cdot})\|^2_{L^2(\mathbb{R})}\leqslant\delta^{-2n}\varkappa.$$

Combining this inequality with the above estimate $\int_{\mathbb{R}\setminus\bigcup_1^n I_k}|\widehat{f}|^2\leqslant 4(n+1)(\frac{4e}{\delta})^{2n}\varkappa^2$, we get

$$\|\widehat{f}(\cdot)\prod_{j=1}^n\theta_\tau(\cdot-\lambda_j)\|_{L^2(\mathbb{R})}\leqslant\left\{4(n+1)(4e)^{2n}+1\right\}^{\frac12}\delta^{-n}\varkappa.$$

It remains to notice that if $\delta=\frac{1}{8n(n+1)}$, then the right-hand side is less than $\{20(n+1)\}^{2n}\varkappa$.

§3.2. Local behavior of functions from the class $EP^n_{\mathrm{loc}}(\tau,\varkappa)$

We start with recalling a well-known lemma on differential equations.

Lemma 3.2. *Let $D = \prod_{j=1}^{n}(e^{2\pi i\lambda_j x}\frac{d}{dx}e^{-2\pi i\lambda_j x})$ be a differential operator of order $n \geqslant 1$ ($\lambda_1, \dots, \lambda_n \in \mathbb{R}$). Let a function f satisfy the equation $Df = g$ on an interval $J = [a, b]$. Then there exists an exponential polynomial p with spectrum $\lambda_1, \dots, \lambda_n$ such that $\sup_{x\in J}|f(x) - p(x)| \leqslant |J|^n \frac{1}{|J|}\int_J |g(t)|\,dt$.*

Proof. Since the space of exponential polynomials with spectrum $\lambda_1, \dots, \lambda_n$ is precisely the space of solutions of the homogeneous equation corresponding to the operator D, it suffices to find a solution φ of the equation $D\varphi = g$ such that $|\varphi(x)| \leqslant |J|^n \frac{1}{|J|}\int_J |g(t)|\,dt$ for all $x \in J$. As usual, one can prove by induction on n that $\varphi = \prod_{j=1}^{n}(e^{2\pi i\lambda_j x}\mathcal{J}e^{-2\pi i\lambda_j x})g$ fits, where $\mathcal{J}$ is the integral operator defined by $(\mathcal{J}h)(x) = \int_a^x h(t)\,dt$, and the lemma is proved.

Now let $f \in EP^n_{\text{loc}}(\tau, \varkappa)$. As shown in §1, there exist $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that $\|\widehat{f}(\cdot)\prod_{j=1}^{n}\theta_\tau(\cdot - \lambda_j)\|_{L^2(\mathbb{R})} \leqslant \varkappa'$ with $\varkappa' \overset{\text{def}}{=} \{20(n+1)\}^{2n}\varkappa$. We put $I_k \overset{\text{def}}{=} (\lambda_k - \frac{1}{\tau}, \lambda_k + \frac{1}{\tau})$, $\widetilde{I}_k \overset{\text{def}}{=} (\lambda_k - \frac{2}{\tau}, \lambda_k + \frac{2}{\tau})$, $E_0 \overset{\text{def}}{=} \mathbb{R} \setminus \bigl(\bigcup_{k=1}^{n} I_k\bigr)$,and $E_k \overset{\text{def}}{=} I_k \setminus \bigl(\bigcup_{j=1}^{k-1} I_j\bigr)$. The sets $E_0, E_1, \dots, E_n$ form a partition of the real line. Now, the function f can be decomposed into the sum $f = \sum_{k=0}^{n}(\widehat{f}\chi_{E_k})^\vee \overset{\text{def}}{=} \sum_{k=0}^{n} f_k$. The spectrum of the function f_k lies in the set $E_k \subset I_k$ ($k = 1, \dots, n$). Moreover,

$$\varkappa'^2 \geqslant \|\widehat{f}(\cdot)\prod_{j=1}^{n}\theta_\tau(\cdot - \lambda_j)\|^2_{L^2(\mathbb{R})} = \sum_{k=0}^{n}\|\widehat{f}_k(\cdot)\prod_{j=1}^{n}\theta_\tau(\cdot - \lambda_j)\|^2_{L^2(\mathbb{R})}.$$

Consequently,

$$\left\|\widehat{f}_k(\cdot)\prod_{j=1}^{n}\theta_\tau(\cdot - \lambda_j)\right\|_{L^2(\mathbb{R})} \leqslant \varkappa'$$

for each $k = 0, \dots, n$. But $\prod_{j=1}^{n}\theta_\tau(\cdot - \lambda_j) \equiv 1$ on E_0, therefore $\|f_0\|^2_{L^2(\mathbb{R})} \leqslant \varkappa'$. Now let $1 \leqslant k \leqslant n$. On the interval I_k we may keep only the factors $\theta_\tau(\cdot - \lambda_j)$ with $\lambda_j \in \widetilde{I}_k$ because all the other factors are identically equal to 1. Let n_k denote the number of points λ_j lying in $\widetilde{I}_k$. Obviously, $1 \leqslant n_k \leqslant n$. We define a differential operator D_k of order n_k by $D_k \overset{\text{def}}{=} \prod_{\lambda_j\in\widetilde{I}_k}(e^{2\pi i\lambda_j x}\frac{d}{dx}e^{-2\pi i\lambda_j x})$. The function f_k has a bounded spectrum, therefore, it is differentiable infinitely many times on $\mathbb{R}$. Setting $g_k \overset{\text{def}}{=} D_k f_k$, we have $\hat{g}_k(\cdot) = (2\pi i)^{n_k}\prod_{\lambda_j\in\widetilde{I}_k}(\cdot - \lambda_j)\widehat{f}_k(\cdot)$. Since $|\lambda - \lambda_j| \leqslant \frac{3}{\tau}\theta_\tau(\lambda - \lambda_j)$ for every $\lambda \in I_k$, $\lambda_j \in \widetilde{I}_k$, we see that

$$\begin{aligned}\|g_k\|_{L^2(\mathbb{R})} = \|\hat{g}_k(\cdot)\|_{L^2(\mathbb{R})} &\leqslant \left(\frac{6\pi}{\tau}\right)^{n_k}\left\|\widehat{f}_k(\cdot)\prod_{\lambda_j\in\widetilde{I}_k}\theta_\tau(\cdot - \lambda_j)\right\|_{L^2(\mathbb{R})} \\ &= \left(\frac{6\pi}{\tau}\right)^{n_k}\left\|\widehat{f}_k(\cdot)\prod_{j=1}^{n}\theta_\tau(\cdot - \lambda_j)\right\|_{L^2(\mathbb{R})} \leqslant \left(\frac{6\pi}{\tau}\right)^{n_k}\varkappa'.\end{aligned}$$

Let $J \subset \mathbb{R}$ be an arbitrary interval of length $|J| \leqslant M\tau$ ($M \geqslant 1$). In accordance with Lemma 3.2, there exists an exponential polynomial $p_k^{(J)}$ with spectrum consisting of the points $\lambda_j \in$

$\widetilde{I}_k$, such that $|f_k(x) - p_k^{(J)}(x)| \leqslant |J|^{n_k} \frac{1}{|J|} \int_J |g_k(t)|\,dt$ for all $x \in J$. Let $\tilde{g}_k \stackrel{\text{def}}{=} \tau^{n_k} g_k$. Then the right-hand side can be estimated from above by $M^{n_k} \frac{1}{|J|} \int_J |\tilde{g}_k(t)|\,dt \leqslant M^n \mathfrak{M}\tilde{g}_k(x)$, where $\mathfrak{M}h(x) \stackrel{\text{def}}{=} \sup_{J: x\in J} \frac{1}{|J|} \int_J |h(t)|\,dt$ is the Hardy–Littlewood maximal function. The well-known estimate of the norm of the maximal function operator in $L^2(\mathbb{R})$ yields

$$\|\mathfrak{M}\tilde{g}_k\|_{L^2(\mathbb{R})} \leqslant 4\|\tilde{g}\|_{L^2(\mathbb{R})} = 4\tau^{n_k}\|g_k\|_{L^2(\mathbb{R})} \leqslant 4(6\pi)^{n_k}\varkappa'.$$

If $p^{(J)} \stackrel{\text{def}}{=} \sum_{k=1}^n p_k^{(J)}$, then $p^{(J)}$ is an exponential polynomial with spectrum $\lambda_1, \dots, \lambda_n$, and for every $x \in J$ we have

$$\begin{aligned} |f(x) - p^{(J)}(x)| &\leqslant |f_0(x)| + M^n \sum_{k=1}^n \mathfrak{M}\tilde{g}_k(x) \\ &\leqslant M^n \left(|f_0(x)| + \sum_{k=1}^n \mathfrak{M}\tilde{g}_k(x) \right) \stackrel{\text{def}}{=} M^n \Phi(x). \end{aligned}$$

The norm of the "error function" $\Phi(x)$ satisfies the inequality

$$\begin{aligned} \|\Phi\|_{L^2(\mathbb{R})} &\leqslant \varkappa' + \sum_{k=1}^n 4(4\pi)^{n_k}\varkappa' \leqslant (4n+1)(6\pi)^n \varkappa' \\ &= (4n+1)(6\pi)^n\{20(n+1)\}^{2n}\varkappa \leqslant \{200(n+1)\}^{2n}\varkappa, \end{aligned}$$

and this proves the desired statement.

§3.3. Expanding the estimate

First of all, we shall reformulate the Turan lemma for the setting where the smallness of an exponential polynomial is understood in the L^2-sense instead of the L^∞-sense as it was in Chapter 1.

Lemma 3.3. *Let $p(x)$ be an exponential polynomial of order n with pure imaginary exponents*, $I \subset \mathbb{R}$ an interval, and $e \subset I$ a set of positive Lebesgue measure. Then

$$\left(\int_I |p(x)|^2\,dx \right)^{\frac{1}{2}} \leqslant \left\{ \frac{A\mu(I)}{\mu(e)} \right\}^{n-\frac{1}{2}} \left(\int_e |p(x)|^2\,dx \right)^{\frac{1}{2}},$$

where A is an absolute constant.

Proof. By the Chebyshev inequality, $|p(x)|^2 \leqslant \frac{2}{\mu(e)} \int_e |p(t)|^2\,dt$ on a subset $e' \subset e$ of measure $\mu(e') \geqslant \frac{1}{2}\mu(e)$. An application of the "L^∞-Turan lemma" to the set e' leads to the estimate

$$\sup_{x\in I} |p(x)|^2 \leqslant \left\{ \frac{A\mu(I)}{\mu(e')} \right\}^{2(n-1)} \frac{2}{\mu(e)} \int_e |p(t)|^2\,dt.$$

Integrating this inequality over I and replacing $\mu(e')$ by $\frac{1}{2}\mu(e)$, we obtain the statement of the lemma. It should be noted that this result was essentially proved in Chapter 2. Here we recall it for convenience of the references.

Now let $f \in EP^n_{\text{loc}}(\tau, \varkappa)$, let $\lambda_1, \dots, \lambda_n$ be an essential spectrum of f, and Φ an "error function" corresponding to this spectrum. Let $E \subset \mathbb{R}$ be a measurable set of finite measure. Suppose that $\int_{\mathbb{R}\setminus E} |f|^2 \leqslant \widetilde{\varkappa}^2$. (The reader may regard $\widetilde{\varkappa}$ to be comparable with $\varkappa$, though from the formal point of view this is inessential). Fixing $\gamma \in (0,1)$ and $M > 1$, we decompose the real line into intervals of length $M\tau$ and take one of them (say, J). Two cases are possible:

1) $\mu(J \setminus E) \geqslant \gamma\mu(J)$;

2) $\mu(J \setminus E) < \gamma\mu(J)$.

In case 1 we call the interval J *white* and in case 2 we call it *black*. If we think γ to be very small, we can say that every black interval consists mostly of points of E, while every white one contains a noticeable piece of $\mathbb{R} \setminus E$.

Let J be white. Denote by p an exponential polynomial with spectrum $\lambda_1, \dots, \lambda_n$ approximating f on J up to an error $M^n\Phi$. The Minkowski inequality implies

$$\Big(\int_{J\setminus E} |p|^2\Big)^{\frac{1}{2}} \leqslant \Big(\int_{J\setminus E} |f|^2\Big)^{\frac{1}{2}} + M^n\Big(\int_J \Phi^2\Big)^{\frac{1}{2}}.$$

Applying Lemma 3.3, we obtain the inequality

$$\begin{aligned}\Big(\int_J |p|^2\Big)^{\frac{1}{2}} &\leqslant \Big\{\frac{A\mu(J)}{\mu(J\setminus E)}\Big\}^{n-\frac{1}{2}}\Big\{\Big(\int_{J\setminus E} |f|^2\Big)^{\frac{1}{2}} + M^n\Big(\int_J \Phi^2\Big)^{\frac{1}{2}}\Big\} \\ &\leqslant \Big(\frac{A}{\gamma}\Big)^{n-\frac{1}{2}}\Big\{\Big(\int_{J\setminus E} |f|^2\Big)^{\frac{1}{2}} + M^n\Big(\int_J \Phi^2\Big)^{\frac{1}{2}}\Big\};\end{aligned}$$

therefore,

$$\begin{aligned}\Big(\int_J |f|^2\Big)^{\frac{1}{2}} &\leqslant \Big(\int_J |p|^2\Big)^{\frac{1}{2}} + M^n\Big(\int_J \Phi^2\Big)^{\frac{1}{2}} \\ &\leqslant \Big(\frac{A}{\gamma}\Big)^{n-\frac{1}{2}}\Big\{\Big(\int_{J\setminus E} |f|^2\Big)^{\frac{1}{2}} + 2M^n\Big(\int_J \Phi^2\Big)^{\frac{1}{2}}\Big\}.\end{aligned}$$

Squaring and summing over all the white intervals J, we find

$$\begin{aligned}\int_{\bigcup\{\,J: J \text{ is white}\,\}} |f|^2 &\leqslant \Big(\frac{A'}{\gamma}\Big)^{2n-1}\Big\{\int_{\mathbb{R}\setminus E} |f|^2 + M^{2n}\int_{\mathbb{R}} \Phi^2\Big\} \\ &\leqslant \Big(\frac{A''n^2M}{\gamma}\Big)^{2n}(\widetilde{\varkappa}^2 + \varkappa^2),\end{aligned} \tag{$**$}$$

where A' and A'' are absolute constants. (We used the estimate $\|\Phi\|^2_{L^2(\mathbb{R})} \leqslant \{200(n+1)\}^{4n}\varkappa^2$ proved in §2 and the condition $\int_{\mathbb{R}\setminus E} |f|^2 \leqslant \widetilde{\varkappa}^2$). Let E_w and E_b be the parts of E

lying in white and black intervals, respectively. The last inequality can be regarded as an expansion of the estimate of f from $\mathbb{R}\smallsetminus E$ to E_w. Its usefulness depends on the measure of E_w: the more this measure is, the more valuable the expansion is.

Later we shall need the following simple observation relating the measure $\mu(E_w)$ with another characteristic of E, namely, the measure of the set $(E+n\tau)\smallsetminus E$.

Lemma 3.4. *If $\gamma<1/2$ and $|t|\leqslant n\tau$, then*

$$\mu(E\smallsetminus(E+t))\leqslant\mu(E_w)+\left(2\gamma+\frac{2n}{M}\right)\mu(E).$$

Proof. Obviously, $\mu(E\smallsetminus(E+t))\leqslant\mu(E_w)+\mu(E_b\smallsetminus(E+t))$. Let an interval J be black. Then

$$\begin{aligned}\mu((E_b\cap J)\smallsetminus(E+t))&\leqslant\mu(J\smallsetminus(E+t))\leqslant\mu(J\smallsetminus(J+t))+\mu((J+t)\smallsetminus(E+t))\\&\leqslant|t|+\mu(J\smallsetminus E)\leqslant n\tau+\gamma\mu(J)=\left(\gamma+\frac{n}{M}\right)\mu(J).\end{aligned}$$

It remains to notice that the number of black intervals does not exceed $\frac{\mu(E)}{(1-\gamma)\mu(J)}\leqslant 2\frac{\mu(E)}{\mu(J)}$.

Corollary 3.5. *Let $\mu(E\smallsetminus(E+n\tau))=\nu$, $f\in EP^n_{\mathrm{loc}}(\tau,\varkappa)$, $\int_{\mathbb{R}\smallsetminus E}|f|^2\leqslant\widetilde{\varkappa}^2$. There exists a subset $E'\subset E$ of measure $\mu(E')\geqslant\frac{\nu}{2}$ such that*

$$\int_{(\mathbb{R}\smallsetminus E)\cup E'}|f|^2\leqslant\left(\frac{A'''n^3\mu(E)^2}{\nu^2}\right)^{2n}(\widetilde{\varkappa}^2+\varkappa^2).$$

Proof. We set $\gamma\overset{\text{def}}{=}\frac{\nu}{8\mu(E)}$, $M\overset{\text{def}}{=}8n\frac{\mu(E)}{\nu}$. It is immediate from the condition $\mu(E\smallsetminus(E+n\tau))=\nu$ that $\mu(E)\geqslant\nu$, and, therefore, $M>1$. We show that as E' one can take the set E_w corresponding to these values of parameters. Indeed, by Lemma 3.4, we have

$$\begin{aligned}\nu=\mu(E\smallsetminus(E+n\tau))&\leqslant\mu(E_w)+\left(2\gamma+\frac{2n}{M}\right)\mu(E)\\&=\mu(E_w)+\left(\frac{\nu}{4\mu(E)}+\frac{2n\nu}{8n\mu(E)}\right)\mu(E)=\mu(E_w)+\frac{\nu}{2},\end{aligned}$$

and, therefore, $\mu(E_w)\geqslant\frac{\nu}{2}$. On the other hand, $(**)$ implies that

$$\begin{aligned}\int_{(\mathbb{R}\smallsetminus E)\cup E_w}|f|^2&\leqslant\int_{\mathbb{R}\smallsetminus E}|f|^2+\int_{\bigcup\{\,J:J\text{ is white}\,\}}|f|^2\leqslant\widetilde{\varkappa}^2+\left(\frac{A''n^2M}{\gamma}\right)^{2n}(\widetilde{\varkappa}^2+\varkappa^2)\\&=\widetilde{\varkappa}^2+\left(\frac{64A''n^3\mu(E)^2}{\nu^2}\right)^{2n}(\widetilde{\varkappa}^2+\varkappa^2)=\left(\frac{A'''n^3\mu(E)^2}{\nu^2}\right)^{2n}(\widetilde{\varkappa}^2+\varkappa^2)\end{aligned}$$

if A''' is suitably chosen.

Though the statement of Corollary 3.5 looks rather cumbersome, it turns out to be most convenient for applications, to which we pass now.

§3.4. The Zygmund theorem on the unit circle

In 1948, Zygmund proved the following uniqueness theorem.

Theorem. *Let a set* $\Lambda = \{m_k\}_{k=-\infty}^{\infty} \subset \mathbb{Z}$ *satisfy the condition*

$$\sup_{r\neq 0} \operatorname{card}\{\, (k', k'') : m_{k'} - m_{k''} = r \,\} \overset{\text{def}}{=} R(\Lambda) < +\infty. \tag{3.1}$$

If $f \in L^2(\mathbb{T})$ *and* $\operatorname{spec} f \subset \Lambda$, *then*

$$\|f\|^2_{L^2(\mathbb{T})} \leqslant C(E, \Lambda) \int_E |f|^2 \, d\mu, \tag{3.2}$$

where $E \subset \mathbb{T}$ *is an arbitrary set of positive measure and the constant* $C(E, \Lambda)$ *depends on* E *and* Λ *but not on* f.

The conditions on the spectrum Λ sufficient for (3.2) have been weakened by many authors. As far as *I* know, the following result due to Mikheev is the best one at present.

Theorem. (Mikheev, 1975). *If a spectrum* Λ *satisfies the condition*

$$\lim_{\varepsilon\to 0} \sup\left\{ \frac{\int_E |f|^2 \, d\mu}{\|f\|^2_{L^2(\mathbb{T})}} : E \subset \mathbb{T},\ \mu(E) < \varepsilon,\ f \in L^2(\mathbb{T}),\ \operatorname{spec} f \subset \Lambda \right\} = 0,$$

then (3.2) still holds.

In particular, (3.2) holds for an arbitrary S_p-system Λ ($p > 2$) because in this case the Hölder inequality yields

$$\int_E |f|^2 \, d\mu \leqslant \mu(E)^{1-\frac{2}{p}} \left(\int_E |f|^p \, d\mu \right)^{\frac{2}{p}} \leqslant C(\Lambda) \varepsilon^{1-\frac{2}{p}} \|f\|^2_{L^2(\mathbb{T})}.$$

We are going to improve Zygmund's result in other direction, namely, we shall show that the constant $C(E, \Lambda)$ can be chosen depending only on the measure of the set E and the bound $R(\Lambda)$ defined in (3.1). Here is the precise statement.

Theorem 3.6 (the Zygmund theorem). *If a spectrum* Λ *satisfies* (3.1), *then*, for an arbitrarily small $\varepsilon > 0$, in inequality (3.2) we can take

$$C(E, \Lambda) = \exp\left\{ \frac{B(\varepsilon, R(\Lambda))}{\mu(E)^{2+\varepsilon}} \right\},$$

where $B(\varepsilon, R(\Lambda))$ is a constant depending only on ε and $R(\Lambda)$.

Corollary 3.7 (the "logarithm summability" theorem). *Under the same assumptions on* Λ, *if* $f \in L^2(\mathbb{T})$ *and* $\operatorname{spec} f \subset \Lambda$, *then for each* $\varepsilon > 0$

$$\int_{\mathbb{T}} \log^{\frac{1}{2}-\varepsilon} \left(1 + \frac{1}{|f|} \right) d\mu < +\infty$$

unless $f \equiv 0$.

Unfortunately, condition (3.1) plays the same crucial role in our proof as it did in original Zygmund's proof. It remains unclear, for example, whether the estimate of the constant $C(E,\Lambda)$ depending only on the measure of E is possible if Λ is a union of two spectra, each of them satisfying (3.1). As to the sharpness of Theorem 3.6, it is not so bad as one could think: from a result due to Mandelbrojt it follows that for each $\varepsilon > 0$ there exists a spectrum Λ satisfying (3.1) for which the constant $C(E,\Lambda)$ cannot be taken less than $\exp\{\frac{B}{\mu(E)^{1-\varepsilon}}\}$ even if we restrict ourselves to arcs $E \subset \mathbb{T}$.

Proof of Theorem 3.6. First of all we shall show that all the results of §§3.1–3.3 remain valid for the functions of class $L^2_{\text{per}}(\mathbb{R})$, i.e., 1-periodic functions which are square-integrable on the period. The norm of a function f from this class is defined by

$$\|f\|_{L^2_{\text{per}}(\mathbb{R})} \stackrel{\text{def}}{=} \left(\int_0^1 |f(x)|^2\,dx\right)^{\frac{1}{2}}.$$

The arguments in §§3.1 and 3.2 will need no correction if we replace $\|\cdot\|_{L^2(\mathbb{R})}$ by $\|\cdot\|_{L^2_{\text{per}}(\mathbb{R})}$ everywhere (the estimate of the norm of the maximal function operator remains valid in the 1-periodic case) and if we integrate $\widehat{f}$ (understood now as $\widehat{f}(\lambda) \stackrel{\text{def}}{=} \int_0^1 f(x)e^{-2\pi i\lambda x}\,dx$) with respect to the measure $\widehat{\mu} \stackrel{\text{def}}{=} \sum_{k=-\infty}^{\infty}\delta(\cdot - k)$ instead of Lebesgue measure. As usual, we denote by δ the Dirac measure, i.e., the unit mass concentrated at the origin. Now it does not matter whether τ is greater or less than 1 and whether the numbers λ_j are integers or not (though the multiplication by $e^{2\pi i\lambda x}$ does not preserve the class L^2_{per}, the operator $D = e^{2\pi i\lambda x}\frac{d}{dx}e^{-2\pi i\lambda x} = \frac{d}{dx} - 2\pi i\lambda$ is still well-defined and the formula $\widehat{Df}(\cdot) = 2\pi i\widehat{f}(\cdot)(\cdot - \lambda)$ remains true on the support of the measure $\widehat{\mu}$). In §3.3 we need the following additional corrections:

a) we assume $\tau \leqslant 1$;

b) we assume E to be a 1-periodic set such that $\mu_{\text{per}}(E) < 1$ (we denote $\mu_{\text{per}}(E) \stackrel{\text{def}}{=} \mu(E\cap(0,1))$).

Now the reader is invited to find out where the Lebesgue measure μ should be replaced by μ_{per}, taking into account the final correction:

c) in the argument preceding Corollary 3.5 we impose on M an additional restriction, namely, we require that the number $\frac{1}{M\tau}$ be an integer.

This condition ensures that the sets E_w and E_b are 1-periodic automatically.

Thus, Corollary 3.5 remains valid if the number M defined in its proof by $M = 8n\frac{\mu_{\text{per}}(E)}{\nu}$ satisfies the inequality $M\tau \leqslant 1$, for in this case one can choose an $M' \in [M, 2M]$ such that $\frac{1}{M'\tau}$ is an integer; replacing M by M' results only in doubling the constant A'''. If $M\tau > 1$, then we take $M' = \frac{1}{\tau}$. Since $\mu_{\text{per}}(E \setminus (E + n\tau)) = \nu$, we have, at any rate, $\mu_{\text{per}}(E) \geqslant \nu$ and $\mu_{\text{per}}(\mathbb{R}\setminus E) \geqslant \nu$. Let $\gamma' \stackrel{\text{def}}{=} \mu_{\text{per}}(\mathbb{R}\setminus E)$. Now our decomposition of the real line into intervals of length $M'\tau$ is a decomposition into periods, each period being white. Inequality $(**)$ takes the form

$$\int_0^1 |f|^2 \leqslant \left(\frac{A''n^2M'}{\gamma'}\right)^{2n}(\widetilde{\varkappa}^2 + \varkappa^2) \leqslant \left(\frac{A'''n^3\mu_{\text{per}}(E)^2}{\nu^2}\right)^{2n}(\widetilde{\varkappa}^2 + \varkappa^2)$$

since $M' \leqslant M = 8n\frac{\mu_{\mathrm{per}}(E)}{\nu}$ and $\frac{1}{\gamma'} = \frac{1}{\mu_{\mathrm{per}}(\mathbb{R}\setminus E)} \leqslant \frac{2\mu_{\mathrm{per}}(E)}{\nu}$ (the latter inequality can be rewritten as $\mu_{\mathrm{per}}(E)\mu_{\mathrm{per}}(\mathbb{R}\setminus E) \geqslant \frac{\nu}{2}$, which is evident because each of the factors in at least ν and their sum is equal to 1). Thus, we can simply take $E' = \mathbb{R}\setminus E$.

Instead of 1-periodic functions defined on the real line one can consider functions defined on $\mathbb{T}$. In this setting, the functions f_{kt} in the definition of the class $EP^n_{\mathrm{loc}}(\tau,\varkappa)$ should be understood as $f_{kt}(z) \stackrel{\mathrm{def}}{=} f(e^{2\pi ikt}z)$.

In these terms, Corollary 3.5 can be reformulated as follows.

Corollary 3.5′. *Let $\mu(\widetilde{E}\setminus e^{2\pi in\tau}\widetilde{E}) = \nu$, $f \in EP^n_{\mathrm{loc}}(\tau,\varkappa)$, and $\int_{\mathbb{T}\setminus\widetilde{E}}|f|^2 \leqslant \widetilde{\varkappa}^2$. There exists a subset $\widetilde{E}' \subset \widetilde{E}$ of measure $\mu(\widetilde{E}') \geqslant \frac{\nu}{2}$ such that*

$$\int_{(\mathbb{T}\setminus\widetilde{E})\cup\widetilde{E}'}|f|^2 \leqslant \left(\frac{A'''n^3\mu(\widetilde{E})^2}{\nu^2}\right)^{2n}(\widetilde{\varkappa}^2+\varkappa^2).$$

(We have replaced E by E' because a set denoted by E is already present in the statement of Theorem 3.6, and we are going to apply the corollary not to this set but to its complement).

Now, at last, we can proceed to the proof of Theorem 3.6. Essentially, we shall copy the classical proof due to Zygmund, though some technicalities will differ slightly.

Let $f \in L^2(\mathbb{T})$, $\operatorname{spec} f \subset \Lambda$, $E \subset \mathbb{T}$. Repeating the argument of Zygmund, we write $f(z) = \sum_{k=-\infty}^{\infty}\widehat{f}(m_k)z^{m_k}$, and

$$\begin{aligned}\int_E |f|^2\,d\mu &= \int_{\mathbb{T}}\chi_E|f|^2\,d\mu = \int_{\mathbb{T}}\chi_E(z)\left|\sum_{k=-\infty}^{\infty}\widehat{f}(m_k)z^{m_k}\right|^2 d\mu\\ &= \mu(E)\sum_{k=-\infty}^{\infty}|\widehat{f}(m_k)|^2 + \sum_{k'\neq k''}\widehat{\chi}_E(m_{k'}-m_{k''})\widehat{f}(m_{k''})\overline{\widehat{f}}(m_{k'})\\ &= \mu(E)\|f\|^2_{L^2(\mathbb{T})} + \langle Q_E f, f\rangle,\end{aligned}$$

where Q_E is an operator in $L^2(\mathbb{T})$. The matrix of Q_E in the basis $\{z^j\}_{j\in\mathbb{Z}}$ is $(q_{js})_{j,s\in\mathbb{Z}}$, and all q_{js} vanish except those of the form $q_{m_{k''}m_{k'}}$ $k' \neq k''$, which are equal to $\widehat{\chi}_E(m_{k''}-m_{k'})$.

Condition (3.1) imposed on Λ implies that

$$\begin{aligned}\sum_{j,s\in\mathbb{Z}}|q_{js}|^2 &= \sum_{k'\neq k''}|\widehat{\chi}_E(m_{k'}-m_{k''})|^2\\ &= \sum_{r\neq 0}|\widehat{\chi}_E(r)|^2\operatorname{card}\{\,(k',k''): m_{k'}-m_{k''} = r\,\}\\ &\leqslant R(\Lambda)\sum_{r\neq 0}|\widehat{\chi}_E(r)|^2 = R(\Lambda)\mu(E)(1-\mu(E)).\end{aligned}$$

Hence Q_E is a selfadjoint Hilbert–Schmidt operator (the selfadjointness of Q_E follows from the fact that the function χ_E is real-valued, which implies $\widehat{\chi}_E(-r) = \overline{\widehat{\chi}}_E(r)$. We have

$$\|Q_E\| \leqslant \left(\sum_{j,s\in\mathbb{Z}}|q_{js}|^2\right)^{\frac12} \leqslant \sqrt{R(\Lambda)\mu(E)(1-\mu(E))}.$$

If the set E is large enough, more precisely, if $\sqrt{R(\Lambda)\mu(E)(1-\mu(E))} \leqslant \frac{\mu(E)}{2}$, then

$$\int_E |f|^2\, d\mu = \mu(E)\|f\|^2_{L^2(\mathbb{T})} + \langle Q_E f, f\rangle$$
$$\geqslant \mu(E)\|f\|^2_{L^2(\mathbb{T})} - \|Q_E\|\|f\|^2_{L^2(\mathbb{T})} \geqslant \frac{\mu(E)}{2}\|f\|^2_{L^2(\mathbb{T})},$$

whence

$$(*) \qquad \|f\|^2_{L^2(\mathbb{T})} \leqslant \frac{2}{\mu(E)}\int_E |f|^2\, d\mu.$$

Solving the inequality $\sqrt{Rt(1-t)} \leqslant \frac{t}{2}$, we find that $(*)$ holds if $\mu(E) \geqslant \frac{4R(\Lambda)}{4R(\Lambda)+1}$. Now we shall show that $(*)$ holds also in the case $\mu(E) < \frac{4R(\Lambda)}{4R(\Lambda)+1}$ provided that f is orthogonal to some subspace V_E of the space $L^2(\mathbb{T})$ having rather low dimension.

We denote by $\sigma_1, \sigma_2, \dots$ the eigenvalues of the operator Q_E enumerated so that $|\sigma_1| \geqslant |\sigma_2| \geqslant \dots$. Let V_n be the subspace generated by the eigenvectors corresponding to the first n eigenvalues of Q_E. The norm of the restriction of Q_E to the orthogonal complement $V_n^\perp$ of the space V_n is equal to $|\sigma_{n+1}|$. It is well known that $\sum_{s=1}^\infty |\sigma_s|^2 = \sum_{j,s\in\mathbb{Z}} |q_{js}|^2 \leqslant R(\Lambda)\mu(E)(1-\mu(E))$. Hence $|\sigma_{n+1}| \leqslant \frac{1}{n+1}R(\Lambda)\mu(E)(1-\mu(E)) \leqslant \frac{\mu(E)^2}{4}$ provided that $n = n(E) \overset{\text{def}}{=} \big[\frac{4R(\Lambda)(1-\mu(E))}{\mu(E)}\big]$. Thus, putting $V_E \overset{\text{def}}{=} V_{n(E)}$, we see, as before, that $(*)$ holds for every $f \in L^2(\mathbb{T})$ orthogonal to V_E with $\operatorname{spec} f \subset \Lambda$. The above argument for "large" sets E was based actually on the relation $n(E) = 0$, which holds if (and only if) $\mu(E) \geqslant \frac{4R(\Lambda)}{4R(\Lambda)+1}$.

Now let E be an arbitrary set of positive measure; let $\mu(E) \leqslant \frac{4R(\Lambda)}{4R(\Lambda)+1}$. Let $f \in L^2(\mathbb{T})$, $\operatorname{spec} f \subset \Lambda$. The proof of the desired estimate for $\|f\|^2_{L^2(\mathbb{T})}$ in terms of $\int_E |f|^2\, d\mu$ goes as follows.

We put $\widetilde{E} \overset{\text{def}}{=} \mathbb{T} \setminus E$ and $n \overset{\text{def}}{=} \big[\frac{8R(\Lambda)}{\mu(E)}\big]$ (the reason for this choice of n will become clear a bit later). Let $\tau > 0$ be so small that for every $t \in (0, \tau)$ the measure of the set $E_t \overset{\text{def}}{=} \bigcap_{k=0}^n e^{-2\pi ikt}E$ is at least $\frac{\mu(E)}{2}$. For each $k = 0, \dots, n$, we have

$$\int_{E_t} |f_{kt}(z)|^2\, d\mu(z) = \int_{E_t} |f(e^{2\pi ikt}z)|^2\, d\mu(z) \leqslant \int_{e^{-2\pi ikt}E} |f(e^{2\pi ikt}z)|^2\, d\mu(z)$$
$$= \int_E |f|^2\, d\mu \overset{\text{def}}{=} \widetilde{\varkappa}^2.$$

Therefore, for every linear combination $g \overset{\text{def}}{=} \sum_0^n a_k f_{kt}$ with $\sum_{k=0}^n |a_k|^2 = 1$ the following inequality holds:

$$\int_{E_t} |g|^2\, d\mu \leqslant \Big(\sum_{k=0}^n |a_k|^2\Big)\Big(\sum_{k=0}^n \int_{E_t} |f_{kt}|^2\, d\mu\Big) \leqslant (n+1)\widetilde{\varkappa}^2.$$

Now we observe that

$$n(E_t) = \left[\frac{4R(\Lambda)(1-\mu(E_t))}{\mu(E_t)}\right] \leqslant \left[\frac{4R(\Lambda)}{\mu(E_t)}\right] \leqslant \left[\frac{8R(\Lambda)}{\mu(E)}\right] = n$$

since $\mu(E_t) \geqslant \frac{1}{2}\mu(E)$. It follows that we can choose the coefficients a_k in such a way that g is orthogonal to V_{E_t}. Inequality $(*)$ applied to the function g and the set E_t yields

$$\|g\|^2_{L^2(\mathbb{T})} \leqslant \frac{2}{\mu(E_t)}\int_{E_t} |g|^2\,d\mu \leqslant \frac{4}{\mu(E)}(n+1)\widetilde{\varkappa}^2 \overset{\text{def}}{=} \varkappa^2.$$

Since $t \in (0,\tau)$ is arbitrary, this means that $f \in EP^n_{\text{loc}}(\tau,\varkappa)$. Thus, we can apply Corollary 3.5$'$ to estimate the integral of $|f|^2$ over a certain wider set $E_1 \supset E$ by the integral $\int_E |f|^2\,d\mu$ multiplied by some factor $W>0$.

The efficiency of this estimate crucially depends on the number $\nu \overset{\text{def}}{=} \mu(\widetilde{E} \setminus e^{2\pi i n\tau}\widetilde{E})$ which, on one hand, stands (in the power $4n$) in the denominator of the factor W and, on the other hand, serves as an estimate from below for the measure $\mu(E_1 \setminus E)$, so the greater ν is, the better estimate we have (and on the wider set). Unfortunately, the above restriction $\mu(E_t) \geqslant \frac{1}{2}\mu(E)$ for every $t \in (0,\tau)$ hinders us in making ν large: to ensure this condition we must demand $\mu(\widetilde{E} \setminus e^{2\pi i s}\widetilde{E}) \leqslant \frac{\mu(E)}{2n}$ for every $s \in (0, n\tau)$ (at least I cannot devise anything better), and this does not allow us to take $\nu > \frac{\mu(E)}{2n}$. (The sufficiency of the above condition follows from the chain of inequalities

$$\begin{aligned}
\mu(E_t) &= \mu\left(\bigcap_{k=0}^{n} e^{-2\pi i k t}E\right) \\
&= \mu\left(E \setminus \bigcup_{k=1}^{n}(e^{-2\pi i k t}\widetilde{E} \setminus \widetilde{E})\right) \geqslant \mu(E) - \sum_{k=1}^{n}\mu(e^{-2\pi i k t}\widetilde{E} \setminus \widetilde{E}) \\
&= \mu(E) - \sum_{k=1}^{n}\mu(\widetilde{E} \setminus e^{2\pi i k t}\widetilde{E}) \geqslant \mu(E) - n\frac{\mu(E)}{2n} = \frac{\mu(E)}{2},
\end{aligned}$$

since $kt \in (0,n\tau)$ if $t \in (0,\tau)$ and $k = 1,\dots,n$.) One could get $\nu = \frac{\mu(E)}{2n}$ taking $\tau = \frac{\sigma}{n}$, where σ is the smallest number among all $s \in [0,1]$ satisfying $\mu(\widetilde{E} \setminus e^{2\pi i s}\widetilde{E}) \geqslant \frac{\mu(E)}{2n}$. But why does such σ really exist? Let us show that we are lucky at least here. Since the (nonnegative) function $\psi(s) \overset{\text{def}}{=} \mu(\widetilde{E} \setminus e^{2\pi i s}\widetilde{E})$ is continuous and $\psi(0)=0$, it suffices to verify that $\int_0^1 \psi(s)\,ds \geqslant \frac{\mu(E)}{2n}$. It is easy to show that $\int_0^1 \psi(s)\,ds = \mu(E)\mu(\widetilde{E})$; besides, the assumption $\mu(E) \leqslant \frac{4R(\Lambda)}{4R(\Lambda)+1}$ made at the very beginning of this part of the reasoning allows us to write the inequality $\mu(\widetilde{E}) \geqslant \frac{1}{4R(\Lambda)+1}$. It remains to notice that the denominator is less than $2n = 2\left[\frac{8R(\Lambda)}{\mu(E)}\right]$ ($R(\Lambda)$ is an integer!).

Now all the parameters in the statement of Corollary 3.5$'$ are determined, and we may write its conclusion: there exists a set $E_1 \overset{\text{def}}{=} E \cup \widetilde{E}'$ such that

$$1) \qquad \mu(E_1 \setminus E) = \mu(\widetilde{E}') \geqslant \frac{\nu}{2} = \frac{\mu(E)}{4n} = \frac{\mu(E)}{4\left[\frac{8R(\Lambda)}{\mu(E)}\right]} \geqslant \frac{\mu(E)^2}{32R(\Lambda)};$$

$$\text{2)}\qquad \int_{E_1} |f|^2\,d\mu \leqslant \biggl(\frac{A'''n^3\mu(\widetilde{E})^2}{\nu^2}\biggr)^{2n}(\widetilde{\varkappa}^2+\varkappa^2) \overset{(\mu(\widetilde{E})\leqslant 1)}{\leqslant} \biggl(\frac{A'''n^3}{\nu^2}\biggr)^{2n}(\widetilde{\varkappa}^2+\varkappa^2)$$
$$\overset{(\nu=\frac{\mu(E)}{2n})}{=} \biggl(\frac{4A'''n^5}{\mu(E)^2}\biggr)^{2n}\biggl(\frac{4}{\mu(E)}(n+1)+1\biggr)\widetilde{\varkappa}^2$$
$$\overset{(n\leqslant\frac{8R(\Lambda)}{\mu(E)})}{\leqslant} \biggl(\frac{4A^{(\mathrm{IV})}R(\Lambda)^5}{\mu(E)^7}\biggr)^{\frac{16R(\Lambda)}{\mu(E)}}\biggl\{\frac{4}{\mu(E)}\biggl(\frac{8R(\Lambda)}{\mu(E)}+1\biggr)+1\biggr\}\widetilde{\varkappa}^2$$
$$\leqslant \biggl(\frac{BR(\Lambda)}{\mu(E)}\biggr)^{120\frac{R(\Lambda)}{\mu(E)}}\widetilde{\varkappa}^2 = \biggl(\frac{BR(\Lambda)}{\mu(E)}\biggr)^{120\frac{R(\Lambda)}{\mu(E)}}\int_E |f|^2\,d\mu$$

($A^{(\mathrm{IV})}$ and $B>0$ are suitable constants). Now it is clear how to complete the proof of the Zygmund theorem: one simply needs to iterate the above result until the measure of the current expansion of the set E becomes greater than $\frac{4R(\Lambda)}{4R(\Lambda)+1}$. To make this iteration process more elegant, it is convenient to denote by $C(\mu)$ the best possible value of the constant $C>0$ such that the inequality $\|f\|^2_{L^2(\mathbb{T})}\leqslant C\int_E|f|^2\,d\mu$ holds for every function $f\in L^2(\mathbb{T})$ with $\operatorname{spec} f\subset\Lambda$ and every set $E\subset\mathbb{T}$ of measure $\mu(E)\geqslant\mu$ (*a priori* we do not suppose $C(\mu)<+\infty$, though this is clear now). All we have proved by now can be written as follows:

I) $C(\mu)\leqslant\frac{2}{\mu}$ for $\mu\in\bigl[\frac{4R(\Lambda)}{4R(\Lambda)+1},1\bigr]$;

II) $C(\mu)\leqslant\bigl(\frac{BR(\Lambda)}{\mu}\bigr)^{120\frac{R(\Lambda)}{\mu}}C\bigl(\mu+\frac{\mu^2}{32R(\Lambda)}\bigr)$ for $\mu<\frac{4R(\Lambda)}{4R(\Lambda)+1}$.

To derive an explicit expression for the constant $C(\mu)$, we need the following simple observation.

Lemma (on solutions of difference equations). *Let $f\colon(0,1)\to[0,+\infty)$ be a monotone decreasing function and $\mu_*\in(0,1)$.* Let a function $\psi\colon(0,1)\to[0,+\infty]$, bounded on $[\mu_*,1)$, satisfy the difference inequality $\frac{\psi(\mu)-\psi(\mu+d(\mu))}{d(\mu)}\leqslant f(\mu)$, where $\mu\in(0,\mu_*)$ and the function $d\colon(0,\mu_*)\to(0,+\infty)$ meets the next two natural restrictions:

1) if $\mu\in(0,\mu_*)$, then $\mu+d(\mu)\in(0,1)$;

2) if $\nu>0$, then $\inf_{\mu\in[\nu,\mu_*)}d(\mu)>0$.

If, in addition, for every $\mu\in(0,\mu_*)$

3) $f(\mu)\leqslant Pf(\mu+d(\mu))$ with some constant $P>0$, then $\psi(\mu)\leqslant\sup_{[\mu_*,1)}\psi+P\int_\mu^1 f(x)\,dx$.

The proof of this lemma is shorter than its statement. Starting with a number $\mu_0=\mu$ we construct an increasing sequence $\mu_0<\mu_1<\mu_2<\dots$ by setting $\mu_{k+1}\overset{\mathrm{def}}{=}\mu_k+d(\mu_k)$. Properties 1) and 2) imply that this sequence will stop at some term $\mu_s\in[\mu_*,1)$. Then

$$\psi(\mu)=\psi(\mu_s)+\sum_{k=0}^{s-1}\{\psi(\mu_k)-\psi(\mu_{k+1})\}.$$

It remains to notice that $\psi(\mu_s)\leqslant\sup_{[\mu_*,1)}\psi$, and each of the summands of the last sum satisfy

$$\psi(\mu_k)-\psi(\mu_{k+1})=\psi(\mu_k)-\psi(\mu_k+d(\mu_k))\leqslant d(\mu_k)f(\mu_k)$$
$$\leqslant Pd(\mu_k)f(\mu_{k+1})\leqslant P\int_{\mu_k}^{\mu_{k+1}}f(x)\,dx,$$

because f is monotone decreasing.

Now we set $\mu_*=\frac{4R(\Lambda)}{4R(\Lambda)+1}$ and $\psi(\mu)\stackrel{\text{def}}{=}\log C(\mu)$. By virtue of condition II), we have

$$\psi(\mu)-\psi(\mu+\frac{\mu^2}{32R(\Lambda)})\leqslant 120\frac{R(\Lambda)}{\mu}\log\frac{BR(\Lambda)}{\mu}\leqslant\frac{B_\varepsilon}{\mu^{1+\varepsilon}},$$

where $\varepsilon>0$ is arbitrarily small and $B_\varepsilon>0$ depends only on ε and $R(\Lambda)$. Setting $d(\mu)\stackrel{\text{def}}{=}\frac{\mu^2}{32R(\Lambda)}$ (conditions 1) and 2) are clearly satisfied), we find

$$\frac{\psi(\mu)-\psi(\mu+d(\mu))}{d(\mu)}\leqslant 32R(\Lambda)\frac{B_\varepsilon}{\mu^{3+\varepsilon}}\stackrel{\text{def}}{=}f(\mu).$$

The estimate $f(\mu)\leqslant Pf(\mu+d(\mu))$ holds, for example, with $P=2^{3+\varepsilon}$; applying the last lemma, we conclude that $\psi(\mu)$ grows near the origin not faster than $\frac{1}{\mu^{2+\varepsilon}}$, which proves the theorem.

§3.5. The Zygmund theorem on the real line

The most direct analogue of (3.1) on the real line is the condition

$$\sup_{t\in\mathbb{R}}\mu\big(\Sigma\cap(\Sigma+t)\big)\stackrel{\text{def}}{=}R(\Sigma)<+\infty. \tag{3.1$'$}$$

But it is easy to understand that (3.1$'$) implies that Σ is of finite measure (it suffices to take $t=0$). This case has already been fully investigated in Chapter 2.

It is worth recalling, however, that Zygmund himself proved his theorem for Fourier series with the Hadamard gaps. From many points of view, a proper analogue of the Hadamard lacunarity condition on the real line is the condition

$$\sup_{k\in\mathbb{Z}}\mu(\Sigma\cap I_k)\stackrel{\text{def}}{=}D(\Sigma)<+\infty, \tag{3.3}$$

where the intervals

$$I_k\stackrel{\text{def}}{=}\begin{cases}\big[-2^{|k|},-2^{|k|-1}\big), & k<0,\\ [-1,1), & k=0,\\ \big[2^{k-1},2^k\big), & k>0\end{cases}$$

form a "diadic decomposition" of the real line $\mathbb{R}$. Such spectra Σ satisfy the following condition, which is somewhat weaker than (3.1$'$): there exists a partition $\Sigma=\bigcup_{j\in\mathbb{Z}}\Sigma_j$ such that

$$\sup_{j\in\mathbb{Z}}\mu(\Sigma_j)<+\infty,\qquad \sup_{t\in\mathbb{R}}\mu\Big(\bigcup_{j'\neq j''}\big(\Sigma_{j'}\cap(\Sigma_{j''}+t)\big)\Big)\stackrel{\text{def}}{=}\widetilde{R}(\Sigma)<+\infty. \tag{3.4}$$

That (3.3) implies (3.4) will be verified later on; now we shall show that from (3.4) it follows that the estimate

$$\|f\|^2_{L^2(\mathbb{R})} \leqslant \exp\exp\{B\mu(E)\}\int_{\mathbb{R}\setminus E}|f|^2\,d\mu$$

is valid for every function $f \in L^2(\mathbb{R})$ with $\operatorname{spec} f \subset \Sigma$ and every set $E \subset \mathbb{R}$ of finite Lebesgue measure. (The constant B depends only on upper bounds in (3.4).) In accordance with the partition $\Sigma = \cup_{j\in\mathbb{Z}}\Sigma_j$, we decompose f into the sum $f = \sum_{j\in\mathbb{Z}}\big(\widehat{f}\chi_{\Sigma_j}\big)^\vee \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}} f_j$. We have

$$\int_E |f|^2\,d\mu = \int_E\Big|\sum_{j\in\mathbb{Z}} f_j\Big|^2\,d\mu = \sum_{j\in\mathbb{Z}}\int_E |f_j|^2\,d\mu + \sum_{j'\neq j''}\int_E f_{j'}\overline{f}_{j''}\,d\mu.$$

In Chapter 2 it was shown that

$$\int_E |f_j|^2\,d\mu \leqslant \big(1-e^{-A\mu(E)\mu(\Sigma_j)}\big)\|f_j\|^2_{L^2(\mathbb{R})},$$

therefore, the first sum does not exceed

$$\big(1-e^{-A'\mu(E)}\big)\sum_{j\in\mathbb{Z}}\|f_j\|^2_{L^2(\mathbb{R})} = \big(1-e^{-A'\mu(E)}\big)\|f\|^2_{L^2(\mathbb{R})},$$

where $A' \stackrel{\text{def}}{=} A\sup_{j\in\mathbb{Z}}\mu(\Sigma_j)$. The second sum is equal to

$$\begin{aligned}
&\sum_{j'\neq j''}\int_{\mathbb{R}}\chi_E(x)\bigg(\int_{\Sigma_{j'}}\widehat{f}(\lambda')e^{2\pi i\lambda' x}\,d\lambda'\bigg)\overline{\bigg(\int_{\Sigma_{j''}}\widehat{f}(\lambda'')e^{2\pi i\lambda'' x}\,d\lambda''\bigg)}\,dx\\
&\quad= \sum_{j'\neq j''}\iiint\limits_{\substack{x\in\mathbb{R}\\ \lambda'\in\Sigma_{j'},\lambda''\in\Sigma_{j''}}}\chi_E(x)e^{2\pi i(\lambda'-\lambda'')x}\widehat{f}(\lambda')\overline{\widehat{f}(\lambda'')}\,d\lambda'\,d\lambda''\,dx\\
&\quad= \sum_{j'\neq j''}\iint\limits_{\substack{\lambda'\in\Sigma_{j'}\\ \lambda''\in\Sigma_{j''}}}\hat{\chi}_E(\lambda''-\lambda')\widehat{f}(\lambda')\overline{\widehat{f}(\lambda'')}\,d\lambda'\,d\lambda'' = \langle Q_E f, f\rangle,
\end{aligned}$$

where Q_E is an operator in $L^2(\mathbb{R})$ with the symbol

$$q(\lambda'',\lambda') = \begin{cases}\hat{\chi}_E(\lambda''-\lambda'), & \lambda'\in\Sigma_{j'},\ \lambda''\in\Sigma_{j''},\ j'\neq j'',\\ 0 & \text{otherwise.}\end{cases}$$

(If we denote $P_E f \stackrel{\text{def}}{=} \chi_E f$, $P_{\widehat{\Sigma}} f \stackrel{\text{def}}{=} \big(\widehat{f}\chi_\Sigma\big)^\vee$, we may think the operator Q_E to be the difference between the operator $P_{\widehat{\Sigma}}P_E P_{\widehat{\Sigma}}$ and its "diagonal" part $\sum_{j\in\mathbb{Z}} P_{\widehat{\Sigma}_j}P_E P_{\widehat{\Sigma}_j}$, like we did in §3.4). But

$$\begin{aligned}
\iint_{\mathbb{R}^2}|q(\lambda'',\lambda')|^2\,d\lambda'\,d\lambda'' &= \int_{\mathbb{R}}\big|\hat{\chi}_E(t)\big|^2\mu\big(\cup_{j'\neq j''}\big(\Sigma_{j'}\cap(\Sigma_{j''}+t)\big)\big)\,dt\\
&\leqslant \sup_{t\in\mathbb{R}}\mu\big(\cup_{j'\neq j''}\big(\Sigma_{j'}\cap(\Sigma_{j''}+t)\big)\big)\int_{\mathbb{R}}\big|\hat{\chi}_E(t)\big|^2 = \widetilde{R}(\Sigma)\mu(E),
\end{aligned}$$

and again we see that Q_E is a selfadjoint Hilbert–Schmidt operator whose norm $\|Q_E\|$ satisfies

$$\|Q_E\| \leqslant \left(\iint_{\mathbb{R}^2} \left|q(\lambda'', \lambda')\right|^2 d\lambda'\, d\lambda''\right)^{1/2} \leqslant \sqrt{\widetilde{R}(\Sigma)\mu(E)}.$$

If $\mu(E) \leqslant \mu_*$ with μ_* determined by the equation $\sqrt{\widetilde{R}(\Sigma)\mu_*} = \frac{1}{2}e^{-A'\mu_*}$, then

$$(*) \qquad \int_E |f|^2\, d\mu \leqslant \left(1 - \frac{1}{2}e^{-A'\mu(E)}\right)\|f\|^2_{L^2(\mathbb{R})} \quad \text{and} \quad \|f\|^2_{L^2(\mathbb{R})} \leqslant 2e^{A'\mu(E)} \int_{\mathbb{R}\setminus E} |f|^2\, d\mu.$$

If $\mu(E) > \mu_*$, then we can prove, by the same way as in §4, that $(*)$ is valid for all the functions $f \in L^2(\mathbb{R})$ with $\operatorname{spec} f \subset \Sigma$ that are orthogonal to some subspace $V_E \subset L^2(\mathbb{R})$ of dimension $\dim V_E = n(E) \overset{\text{def}}{=} \left[4\widetilde{R}(\Sigma)\mu(E)e^{2A'\mu(E)}\right]$.

For an arbitrary set $E \subset \mathbb{R}$ of finite measure satisfying $\mu(E) > \mu_*$, we put $n \overset{\text{def}}{=} \left[8\widetilde{R}(\Sigma)\mu(E)e^{4A'\mu(E)}\right]$. It τ is so small that for every $t \in (0, n\tau)$ the measure of the set $E_t \overset{\text{def}}{=} \cup_{k=0}^{n}(E - kt)$ is less than $2\mu(E)$ (this implies the estimate $n(E_t) \leqslant n$), then we again notice that the function f belongs to $EP^n_{\text{loc}}(\tau, \varkappa)$ with $\varkappa^2 = 2e^{A'\mu(E)}(n+1)\widetilde{\varkappa}^2$, where $\widetilde{\varkappa} \overset{\text{def}}{=} \int_{\mathbb{R}\setminus E} |f|^2\, d\mu$ ($f \in L^2(\mathbb{R})$, $\operatorname{spec} f \subset \Sigma$ as above).

Let τ be the smallest positive number satisfying $\mu\big(E \setminus (E + n\tau)\big) = \frac{\mu(E)}{n} \overset{\text{def}}{=} \nu$. Applying Corollary 3.5, we see that there exists a subset $E' \subset E$ such that $\mu(E') \geqslant \frac{\nu}{2} \geqslant \frac{1}{16\widetilde{R}(\Sigma)}e^{-4A'\mu(E)}$, and

$$\begin{aligned}
\int_{(\mathbb{R}\setminus E)\cup E'} |f|^2 &\leqslant \left(\frac{A'''n^3\mu(E)^2}{\nu^2}\right)^{2n}(\widetilde{\varkappa}^2 + \varkappa) \\
&\leqslant \left\{A^{(\mathrm{IV})}\widetilde{R}(\Sigma)^5\mu(E)^4e^{20A'\mu(E)}\right\}^{8\widetilde{R}(\Sigma)\mu(E)e^{4A'\mu(E)}}\left\{2e^{A'\mu(E)}\left(e^{4A'\mu(E)} + 1\right)\right\}\int_{\mathbb{R}\setminus E} |f|^2 \\
&\leqslant \exp\exp\{B\mu(E)\}\int_{\mathbb{R}\setminus E} |f|^2\, d\mu
\end{aligned}$$

if the constants $A^{(\mathrm{IV})}$ and B are suitably chosen (we have used the fact that $\mu(E) \geqslant \mu_* > 0$).

Denoting by $C(\mu)$ the best possible constant in the inequality

$$\|f\|^2_{L^2(\mathbb{R})} \leqslant C(\mu)\int_{\mathbb{R}\setminus E} |f|^2\, d\mu \qquad (\mu > 0, \quad \mu(E) \leqslant \mu, \quad f \in L^2(\mathbb{R}), \quad \operatorname{spec} f \subset \Sigma),$$

we obtain the properties

I) $C(\mu) \leqslant 2e^{A'\mu}$ for $0 < \mu < \mu_*$;

II) $C(\mu) \leqslant e^{e^{B\mu}} C\left(\mu - \frac{1}{16\widetilde{R}(\Sigma)}e^{-4A'\mu}\right)$ for $\mu \geqslant \mu_*$,

whence it is not hard to derive the desired estimate

$$C(\mu) \leqslant \exp\exp\{B(\Sigma)\mu\}.$$

It remains to show that for every spectrum Σ sparse in the sense of (3.3), there exists a partition $\Sigma = \cup_{j\in\mathbb{Z}}\Sigma_j$ satisfying (3.4). Let $k > 0$. Well-known estimates of the one-sided maximal function on the real line imply that the half of the interval I_k that is nearest to the origin contains a point ξ_k such that

$$\mu\big(\Sigma \cap I_k \cap [\xi_k, \xi_k + s]\big) \leqslant \frac{2\mu(\Sigma \cap I_k)}{\mu(I_k)} s \leqslant 2^{2-k} D(\Sigma) s$$

for all $s > 0$. Similarly, if $k < 0$, then we define the point ξ_k, again from the half of I_k that is nearest to the origin, by the condition

$$\mu\big(\Sigma \cap I_k \cap [\xi_k - s, \xi_k]\big) \leqslant \frac{2\mu(\Sigma \cap I_k)}{\mu(I_k)} s \leqslant 2^{2-|k|} D(\Sigma) s \qquad (s > 0).$$

Let, finally, $\xi_0 = 0$. We introduce

$$\Sigma_j \stackrel{\text{def}}{=} \begin{cases} [\xi_{j-1}, \xi_j) \cap \Sigma, & j > 0, \\ [\xi_j, \xi_{j+1}) \cap \Sigma, & j < 0. \end{cases}$$

It is clear that $\mu(\Sigma_j) \leqslant 2D(\Sigma)$. Now, for $t \in \mathbb{R}$, let j_0 be the smallest positive integer such that $2^{j_0} \geqslant |t|$. Let us take an arbitrary interval $[a, b]$ of length $|t|$ centered at a point lying to the right from the origin. How can it happen that its right end lies in Σ_j while the left one lies in some Σ_m with $m < j$? Obviously, this occurs only if $b \in \Sigma_j \cap [\xi_{j-1}, \xi_{j-1} + |t|)$. Since $b \geqslant \frac{|t|}{2}$, we have $j > 0$. If $j \geqslant j_0 + 3$, then the distance between the point ξ_{j-1} and the right end of the interval I_{j-1} is greater than $2^{j-3} \geqslant 2^{j_0} \geqslant |t|$, and, therefore, the latter set coincides with $\Sigma \cap I_{j-1} \cap [\xi_{j-1}, \xi_{j-1} + |t|)$. On the other hand, if $0 < j < j_0 - 2$, then all points from Σ_j lie to the left of the point $\frac{|t|}{2}$ (in this case from the definition of j_0 it follows that $2^{j-1} < |t|$ and, consequently, $\xi_j < 2^j \leqslant 2^{j_0-2} < \frac{|t|}{2}$). Therefore, the measure of the set of right ends (and, of course, the measure of the set of left ends, whose value is the same) of all intervals $[a, b]$ with centers to the right of the origin and such that a and b lie in different Σ_j, does not exceed

$$\begin{aligned} &\sum_{j_0-1\leqslant j\leqslant j_0+2} \mu(\Sigma_j) + \sum_{j=j_0+3}^{\infty} \mu\big(\Sigma \cap I_{j-1} \cap [\xi_{j-1}, \xi_{j-1} + |t|)\big) \\ &\qquad \leqslant 4 \times 2D(\Sigma) + \sum_{j=j_0+3}^{\infty} 2^{3-j} D(\Sigma)|t| = 8D(\Sigma) + 2D(\Sigma)\frac{|t|}{2^{j_0}} < 10D(\Sigma). \end{aligned}$$

Considering similarly the intervals with centers to the left of the origin, we conclude that the second supremum in (3.4) does not exceed $20D(\Sigma)$. That's all!

Chapter 4. Some remarks and supplements

§4.1. Estimates of exponential polynomials with complex exponents

Let $p(t)=\sum_{k=1}^{n} c_k e^{\lambda_k t}$ be an exponential polynomial of order n. The same reasoning as we used to prove the classical Turan lemma (see Chapter 1, §1) shows that, $J\subset I\subset\mathbb{R}$ being two intervals,

$$\sup_{t\in I}|p(t)|\leqslant e^{|I|\max|\operatorname{Re}\lambda_k|}\left\{\frac{4e\mu(I)}{\mu(J)}\right\}^{n-1}\sup_{t\in J}|p(t)|$$

(essentially, this was already mentioned when we used the Turan lemma to estimate the constant a in the Hadamard factorization $p(z)=ce^{az}Q(z)R(z)$).

Now we are going to show how the same statement can be obtained for an arbitrary subset $E\subset I$ of positive measure.

Proposition 4.1. *Under the same conditions on* $p(t)$,

$$\sum_{t\in I}|p(t)|\leqslant e^{|I|\max|\operatorname{Re}\lambda_k|}\left\{\frac{A\mu(I)}{\mu(E)}\right\}^{n-1}\sup_{t\in E}|p(t)|$$

(here E is a measurable subset of I, A is a universal constant).

Proof. We dwell on the main ideas only. Without loss of generality, $I=\left[-\frac{1}{2},\frac{1}{2}\right]$. The main difficulty is that the Langer lemma, which provided us with quite sharp information on the local distribution of zeros of an exponential polynomial, cannot be applied now. The following simple observations based on combining the Turan lemma for two intervals with the Jensen inequality (or Schwarz's lemma, if the reader wishes) partly compensate for this lack.

Lemma 4.2. *Let* $p(t)=\sum_{k=1}^{n} c_k e^{\lambda_k t}$ $(c_k\in\mathbb{C},\ \lambda_k\in\mathbb{C})$ *be an exponential polynomial of order* n, $\lambda\stackrel{\text{def}}{=}\max|\lambda_k|$. *Then the number of zeros of* $p(z)$ *inside each disk of radius* $r>0$ *does not exceed* $4(n-1)+7\lambda r$.

Proof. Let D_r be a disk of radius r centered at z_0, and D_R a larger disk of radius $R>r$ centered at the same point. We put $m\stackrel{\text{def}}{=}\max_{z\in D_r}|p(z)|$, $M\stackrel{\text{def}}{=}\max_{z\in D_R}|p(z)|$.

Considering the restrictions of $p(z)$ to all the intervals

$$I_\zeta\stackrel{\text{def}}{=}\{\,z_0+t\zeta : t\in[-r,R]\,\}\supset J_\zeta\stackrel{\text{def}}{=}\{\,z_0+t\zeta : t\in[-r,r]\,\}\qquad(\zeta\in\mathbb{T})$$

and applying the Turan lemma, we get

$$M\leqslant\left\{\frac{4e(R+r)}{2r}\right\}^{n-1}e^{\lambda(R+r)}m.$$

On the other hand, for each zero ξ of $p(z)$ lying in D_r, the maximum the absolute value of the corresponding Blaschke factor $B_\xi(z)\stackrel{\text{def}}{=}\frac{R(z-\xi)}{R^2-\overline{(\xi-z_0)}(z-z_0)}$ in the disk D_r is equal to $\frac{2rR}{R^2+r^2}$. Hence $m\leqslant\left(\frac{2rR}{R^2+r^2}\right)^N M$, where N is the number of zeros of $p(z)$ lying in D_r.

Comparing these two inequalities, we obtain

$$N \leqslant \left\{\log\left(\frac{2rR}{R^2+r^2}\right)\right\}^{-1}\left\{(n-1)\log\frac{4e(R+r)}{2r}+\lambda(R+r)\right\}.$$

Putting $R=6r$ we arrive at $N \leqslant \frac{1}{\log\frac{37}{12}}\{(n-1)\log(14e)+7\lambda r\}$, and the statement follows. The following estimate of the Bernstein lemma type can easily by derived from Lemma 4.2 (This is an analogue of Lemma 1.6 used before).

Lemma 1.6′. *Let* $g(t) \overset{\text{def}}{=} \sum_{k=1}^{n} c_k e^{i\lambda_k t}$, $\lambda_k \in \mathbb{C}$, $\lambda \overset{\text{def}}{=} \max|\lambda_k| > n-1$. *Then*

$$\mu\left\{t \in \left[-\frac{1}{2},\frac{1}{2}\right] : \left|\frac{d}{dt}\log g(t)\right| > y\right\} \leqslant \frac{A\lambda}{y}$$

for every $y>0$, *where* A *is a universal constant.*

The proof of Lemma 1.6′ almost entirely follows the lines of the proof of Lemma 1.6 and is based on the factorization

$$g(z) = ce^{az}\prod_{j\leqslant 11\lambda}(z-z_j)\prod_{j\leqslant 11\lambda}\left(1-\frac{z}{z_j}\right)e^{z/z_j},$$

where the zeros z_j of $p(z)$ are enumerated so that the numbers $|z_j|$ are monotone nondecreasing. Lemma 4.2 implies the inequality $|z_j| \geqslant \frac{j-4(n-1)}{7\lambda} \geqslant \frac{j-4\lambda}{7\lambda} > 1$ (valid for $j>11\lambda$), which can fully replace the inequality $|z_j| \geqslant \pi\frac{j-\lambda}{\lambda}$ used before.

Lemma 1.6′ allows us to reduce the proof of the Turan lemma to the case $\lambda = \operatorname{diam}\operatorname{spec} p \overset{\text{def}}{=} \max_{k',k''}|\lambda_{k'}-\lambda_{k''}| \leqslant n-1$ by constructing by induction a sequence of exponential polynomials $p_n, p_{n-1}, \dots, p_s$ such that

1) $p_n = p$;
2) $\operatorname{ord} p_k = k$ $(k=s,\dots,n)$;
3) $\operatorname{spec} p_{k-1} \subset \operatorname{spec} p_k$ and $\|p_{k-1}\|_\infty \geqslant \delta\|p_k\|_\infty$ $(k=s+1,\dots,n$; $\delta>0$ is an absolute constant);
4) the ratio $\varphi_k \overset{\text{def}}{=} \left|\frac{p_{k-1}}{p_k}\right|$ satisfies the weak type estimate

$$\mu\left\{x \in \left[-\frac{1}{2},\frac{1}{2}\right] : \varphi_k(x) > t\right\} \leqslant \frac{1}{t}$$

for every $t>0$;
5) $\operatorname{diam}\operatorname{spec} p_s \leqslant s-1$.

Unlike the case of pure imaginary exponents, now we must look after the spectrum $\operatorname{spec} p_k$. Fortunately, that leads to no additional trouble since it can easily be seen that the spectra of the auxiliary polynomials $\overline{q}$ and $\underline{q}$ are obtained from the spectrum of $p_k(z)$ merely by excluding one point.

Now let $\operatorname{diam}\operatorname{spec} p \leqslant n-1$. Then $p(z) = e^{\lambda_1 z}\tilde{p}(z)$, where $\widetilde{\lambda}_k \overset{\text{def}}{=} \lambda_k - \lambda_1$, $|\widetilde{\lambda}_k| \leqslant n-1$, and it suffices to show that

$$\sup_{t\in\left[-\frac{1}{2},\frac{1}{2}\right]} |\tilde{p}(t)| \leqslant \left\{\frac{A}{\mu(E)}\right\}^{n-1} \sup_{t\in E} |\tilde{p}(t)|.$$

Again we write the Hadamard factorization

$$\tilde{p}(z) = ce^{az} \prod_{j\leqslant 11\lambda} (z-z_j) \prod_{j>11\lambda} \left(1-\frac{z}{z_j}\right) e^{z/z_j} \overset{\text{def}}{=} ce^{az} Q(z) R(z).$$

The oscillation of the factors ce^{az} and $R(z)$ on the interval $\left[-\frac{1}{2},\frac{1}{2}\right]$ can be majorized by $e^{\text{const}(n-1)}$, as before (of course, to estimate $|a|$ now we should use the Turan lemma instead of evaluating the argument increment on vertical segments). But the information we get from Lemma 4.2 is not sufficient to investigate the set where $Q(z)$ is small: though for every $z \in \left[-\frac{1}{2},\frac{1}{2}\right]$, except, may be, a set of measure $44h$, the Kartan type estimate $|z-z_j| \geqslant \frac{jh}{(n-1)}$ is still valid (as above, z_j are the complex zeros of the polynomial $Q(z)$ enumerated so that the numbers $|z-z_j|$ are monotone nondecreasing), Lemma 4.2 provides us with a nontrivial *a priori* estimate of $|z-z_j|$ only for $j > 4(n-1)$, not for $j > (n-1)$ (as we need). I do not know whether in our case one can prove the inequality $|z-z_j| \geqslant \gamma\frac{j-(n-1)}{(n-1)}$ with some universal constant $\gamma > 0$; we shall pass over this difficulty in another way.

Lemma 4.3. *Let* $\tilde{p}(z) = \sum_{k=1}^{n} c_k e^{\widetilde{\lambda}_k t}$, *where* $|\widetilde{\lambda}_k| \leqslant n-1$ *for each* $k = 1,\dots,n$. *Let* $z \in \mathbb{C}$, *and let* z_j *be complex zeros of* $\tilde{p}(z)$ *enumerated so that the numbers* $|z-z_j|$ *are monotone nondecreasing. Then*

$$\prod_{j=n}^{11(n-1)} \frac{|z-z_j|}{1+|z-z_j|} \geqslant e^{-\text{const}(n-1)},$$

the constant being universal.

Proof. We choose $N \in \{0,\dots,11(n-1)-1\}$ so that $|z-z_N| < 1$ and $|z-z_{N+1}| \geqslant 1$ ($N=0$ means that $|z-z_j| \geqslant 1$ for all $j = 1,2,\dots$). The product $\prod_{j=N+1}^{11(n-1)} \frac{|z-z_j|}{1+|z-z_j|}$ is obviously greater than $2^{-\left(11(n-1)-N\right)} \geqslant 2^{-11(n-1)}$, so the lemma is trivial if $N < n$. Otherwise we estimate the product $\prod_{j=n}^{N} \frac{|z-z_j|}{1+|z-z_j|}$ from below comparing the maximums m and M of the absolute value of the exponential polynomial $\tilde{p}(z)$ over the disks $D_r \overset{\text{def}}{=} \{\,\zeta : |\zeta - z| \leqslant r\,\}$ and $D_1 \overset{\text{def}}{=} \{\,\zeta : |\zeta - z| \leqslant 1\,\}$, where $r \overset{\text{def}}{=} |z-z_n| \in (0,1)$ (r is actually greater than 0, because a nontrivial exponential polynomial of order n cannot have a zero of multiplicity greater than $n-1$).

As above (see the proof of Lemma 4.2), we derive from the Turan lemma that

$$M \leqslant \left\{\frac{4e(1+r)}{2r}\right\}^{n-1} e^{(n-1)(1+r)} m \leqslant (4e^3)^{n-1} \left(\frac{1}{r}\right)^{n-1} m.$$

On the other hand, in the disk D_r the absolute value of the Blaschke factor $B_{z_j}(z)$ corresponding to the root $z_j \in D_1$ does not exceed

$$\frac{|z-z_j|r}{1+|z-z_j|r} \leqslant \begin{cases} 2r, & j=1,\dots,n-1, \\ \frac{4|z-z_j|}{1+|z-z_j|}, & j=n,\dots,N. \end{cases}$$

Thus, $m \leqslant (2r)^{n-1} \prod_{j=n}^{N} \frac{4|z-z_j|}{1+|z-z_j|} M$. It remains to compare the two inequalities obtained above.

It is worth mentioning that the additional factor $e^{|I|\max|\operatorname{Re}\lambda_k|}$ in the Turan type estimate arises because E may be concentrated near one of the endpoints of the interval I. If we want to estimate $p(x)$ at the points $x \in \mathbb{R}$ such that E is "massive enough" on the both sides of x, there is a majoration not involving the exponents λ_k.

Proposition 4.4. (The interpolation Turan lemma). *Let $p(z)$ be an exponential polynomial of order n with arbitrary complex exponents.* Let $E \subset \mathbb{R}$ and $x \in \mathbb{R}$. Then

$$|p(x) \leqslant \left\{\frac{A}{m_E(x)}\right\}^{n-1} \sup_{t\in E} |p(t)|,$$

where $m_E(x) = \min\bigl(M_E^+(x), M_E^-(x)\bigr)$, and

$$M_E^+(x) \overset{\text{def}}{=} \sup_{r>0} \frac{\mu\{E\cap(x-r,x)\}}{r}, \qquad M_E^-(x) \overset{\text{def}}{=} \sup_{r>0} \frac{\mu\{E\cap(x,x+r)\}}{r}$$

are the "one-sided" Hardy–Littlewood maximal functions.

Proof. This is left to the reader as an easy exercise.

In the case of pure imaginary exponents one can replace $m_E(x)$ with $M_E(x) = \max\bigl(M_E^+(x), M_E^-(x)\bigr)$.

§4.2. Functions with spectrum of finite Lebesgue measure

Let $\Sigma \subset \mathbb{R}$ be an open set of finite measure. In Chapter 2 it was shown that for every function $f \in L^2(\mathbb{R})$ with spectrum $\operatorname{spec} f \subset \Sigma$ and for every measurable set $E \subset \mathbb{R}$ of finite measure, the norm $\|f\|_{L^2(\mathbb{R})}$ can be estimated in terms of the integral of $|f|^2$ over $\mathbb{R}\setminus E$, namely,

$$(*) \qquad \|f\|^2_{L^2(\mathbb{R})} \leqslant e^{A\mu(E)\mu(\Sigma)} \int_{\mathbb{R}\setminus E} |f|^2\,d\mu.$$

If Σ is an interval, then this statement may be generalized in two directions.

a) $L^2(\mathbb{R})$ may be replaced by any other Lebesgue function class $L^p(\mathbb{R})$ with $p>0$ (if $p<1$, then the inclusion $\operatorname{spec} f \subset \Sigma$ may be understood, for example, as the possibility of an arbitrarily good approximation of f in the sense of L^p-norm by functions from the Schwartz class S with spectrum in Σ).

b) The condition $\mu(E) < +\infty$ may be replaced with that of "relative density" of $\mathbb{R}\setminus E$ (we say that $\mathbb{R}\setminus E$ is relatively dense if there exist two numbers $D > d > 0$ such that

$\mu(E\cap J)\leqslant D-d$ for every interval J of length D; if $\mu(E)>+\infty$, one can put $D=2\mu(E)$, $d=\mu(E)$).

The corresponding inequality

$$\|f\|^p_{L^p(\mathbb{R})}\leqslant e^{2\pi p|\Sigma|\frac{D^2}{d}}\int_{\mathbb{R}\smallsetminus E}|f|^p\,d\mu$$

is called "the theorem of Logvinenko and Sereda". The proof (based on the methods of theory of analytic functions) is as follows. Without loss of generality $\Sigma=[-s,s]$ (otherwise we consider $fe^{i\lambda x}$ with a properly chosen $\lambda\in\mathbb{R}$ instead of f). Then f is an entire function of exponential type at most $2\pi s$. Consider $f(z)$ on the line $\mathbb{R}+iD$. The relative density condition imposed on $\mathbb{R}\smallsetminus E$ implies that the harmonic measure $\omega_{\mathbb{C}_+}(\mathbb{R}\smallsetminus E,z)$ of the set $\mathbb{R}\smallsetminus E$ with respect to the upper half-plane $\mathbb{C}_+$ and the point $z\in\mathbb{R}+iD$ is at least $\frac{d}{\pi D}\overset{\text{def}}{=}\gamma$ (to prove this, it suffices to observe that the density of the harmonic measure $\omega_{\mathbb{C}_+}(\cdot,z)$ with respect to Lebesgue measure is greater than $\frac{1}{2\pi D}$ on each of the two intervals $(\operatorname{Re}z-D,\operatorname{Re}z)$ and $(\operatorname{Re}z,\operatorname{Re}z+D)$). Therefore, for every $z\in\mathbb{R}+iD$

$$\begin{aligned}\log|f(z)|^p&\leqslant psD+\int_{\mathbb{R}}\log|f(t)|^pP(z,t)\,dt=psD+\int_{\mathbb{R}\smallsetminus E}+\int_E\\&\leqslant psD+\omega_{\mathbb{C}_+}(\mathbb{R}\smallsetminus E,z)\log\int_{\mathbb{R}\smallsetminus E}|f(t)|^pP(z,t)\,dt\\&\quad+\omega_{\mathbb{C}_+}(E,z)\log\int_E|f(t)|^pP(z,t)\,dt.\end{aligned}$$

Hence,

$$\begin{aligned}|f(z)|^p&\leqslant e^{psD}\Bigl(\int_{\mathbb{R}\smallsetminus E}|f(t)|^pP(z,t)\,dt\Bigr)^{\omega_{\mathbb{C}_+}(\mathbb{R}\smallsetminus E,z)}\Bigl(\int_{\mathbb{R}}|f(t)|^pP(z,t)\,dt\Bigr)^{\omega_{\mathbb{C}_+}(E,z)}\\&\leqslant e^{psD}\Bigl(\int_{\mathbb{R}\smallsetminus E}|f(t)|^pP(z,t)\,dt\Bigr)^{\gamma}\Bigl(\int_{\mathbb{R}}|f(t)|^pP(z,t)\,dt\Bigr)^{1-\gamma}.\end{aligned}$$

Thus,

$$\begin{aligned}&\int_{\mathbb{R}}|f(x+iD)|^p\,dx\\&\qquad\leqslant e^{psD}\int_{\mathbb{R}}\Bigl(\int_{\mathbb{R}\smallsetminus E}|f(t)|^pP(x+iD,t)\,dt\Bigr)^{\gamma}\Bigl(\int_{\mathbb{R}}|f(t)|^pP(x+iD,t)\,dt\Bigr)^{1-\gamma}dx.\end{aligned}$$

By the Hölder inequality, the last expression does not exceed

$$\begin{aligned}&e^{psD}\Bigl(\int_{\mathbb{R}}\Bigl(\int_{\mathbb{R}\smallsetminus E}|f(t)|^pP(x+iD,t)\,dt\Bigr)dx\Bigr)^{\gamma}\\&\quad\times\Bigl(\int_{\mathbb{R}}\Bigl(\int_{\mathbb{R}}|f(t)|^pP(x+iD,t)\,dt\Bigr)dx\Bigr)^{1-\gamma}=e^{psD}\Bigl(\int_{\mathbb{R}\smallsetminus E}|f|^p\Bigr)^{\gamma}\Bigl(\int_{\mathbb{R}}|f|^p\Bigr)^{1-\gamma},\end{aligned}$$

and, therefore,

$$\|f(\cdot+iD)\|^p_{L^p(\mathbb{R})} \leqslant e^{psD}\|f(\cdot)\|^{p(1-\gamma)}_{L^p(\mathbb{R})}\biggl(\int_{\mathbb{R}\smallsetminus E}|f|^p\,d\mu\biggr)^{\gamma}.$$

Since f is an entire function, not only can we raise the real line, but lower it as well. We get

$$\|f(\cdot)\|^p_{L^p(\mathbb{R})} \leqslant e^{psD}\|f(\cdot+iD)\|^p_{L^p(\mathbb{R})}.$$

Thus,

$$\|f\|^p_{L^p(\mathbb{R})} \leqslant e^{2psD}\|f\|^{p(1-\gamma)}_{L^p(\mathbb{R})}\biggl(\int_{\mathbb{R}\smallsetminus E}|f|^p\,d\mu\biggr)^{\gamma},$$

whence $\|f\|^p_{L^p(\mathbb{R})} \leqslant e^{2psD/\gamma}\int_{\mathbb{R}\smallsetminus E}|f|^p\,d\mu$, proving the desired statement.

To what extent is such a generalization possible in the case of arbitrary measurable set Σ? We shall restrict ourselves to the question as to whether it is possible to replace the L^2-norm with the L^p-norm in $(*)$. If $0<p<2$, then the answer is affirmative and the corresponding estimate is

$$(*_p) \qquad \|f\|^p_{L^p(\mathbb{R})} \leqslant e^{2Ap\mu(E)\mu(\Sigma)}\int_{\mathbb{R}\smallsetminus E}|f|^p\,d\mu,$$

where A is the same constant as in $(*)$. It should be noted that this inequality is rather sharp, namely, there exists an absolute constant $a>0$ such that for every $p\in(0,2)$ and every two intervals E and Σ, one can find a function $f\in L^p(\mathbb{R})$ with spectrum in Σ satisfying

$$\|f\|^p_{L^p(\mathbb{R})} \geqslant e^{ap\mu(E)\mu(\Sigma)}\int_{\mathbb{R}\smallsetminus E}|f|^p\,d\mu.$$

Proof. Let $f\in S$. We denote by Φ the decreasing rearrangement of $|f|$. (Φ is the function defined on $(0,+\infty)$ by $\Phi^{-1}(y)=\mu(|f|>y)$.) Inequality $(*)$ is equivalent to each of the following estimates:

$$\int_0^t \Phi(x)^2\,dx \leqslant \bigl(1-e^{-A\mu(\Sigma)t}\bigr)\|f\|^2_{L^2(\mathbb{R})} \iff \int_t^\infty \Phi(x)^2\,dx \geqslant e^{-A\mu(\Sigma)t}\|f\|^2_{L^2(\mathbb{R})},$$

for all $t>0$. The first of them implies, by the Hölder inequality, that

$$\int_0^t \Phi(x)^2\,dx \leqslant t\biggl(\frac{1-e^{-A\mu(\Sigma)t}}{t}\biggr)^{p/2}\|f\|^p_{L^2(\mathbb{R})}.$$

To get some information from the second one, we shall apply the following classical lemma on convex functions.

Lemma. *Let $\varphi, \psi\colon (0,+\infty) \to \mathbb{R}$ be nonnegative convex functions such that $\lim_{+\infty} \varphi = \lim_{+\infty} \psi = 0$. If $\varphi(t) \geqslant \psi(t)$ for every $t > 0$, then for every $q \in (0,1)$*

$$\int_t^\infty |\varphi'(x)|^q \, dx \geqslant \int_t^\infty |\psi'(x)|^q \, dx \qquad (t > 0).$$

Putting here $\varphi(t) \overset{\text{def}}{=} \int_t^\infty \Phi(x)^2 \, dx$, $\psi(t) \overset{\text{def}}{=} e^{-A\mu(\Sigma)t} \|f\|^2_{L^2(\mathbb{R})}$, $q \overset{\text{def}}{=} \frac{p}{2}$, we get

$$\begin{aligned} \int_t^\infty \Phi(x)^p \, dx &\geqslant \{A\mu(\Sigma)\}^{p/2} \|f\|^p_{L^2(\mathbb{R})} \int_t^\infty e^{-A\mu(\Sigma)px} \, dx \\ &= \frac{\{A\mu(\Sigma)\}^{p/2-1}}{p} \|f\|^p_{L^2(\mathbb{R})} e^{-A\mu(\Sigma)pt}. \end{aligned}$$

Obviously, $(*_p) \iff \int_0^t \Phi(x)^p \, dx \leqslant \left(e^{2A\mu(\Sigma)pt} - 1\right) \int_t^\infty \Phi(x)^p \, dx$. Taking the above estimates into account, we see that it suffices to prove the inequality

$$t\left(\frac{1 - e^{-A\mu(\Sigma)t}}{t}\right)^{p/2} \leqslant \left(e^{2A\mu(\Sigma)pt} - 1\right) \frac{\{A\mu(\Sigma)\}^{p/2-1}}{p} \|f\|^p_{L^2(\mathbb{R})} e^{-A\mu(\Sigma)pt},$$

which is equivalent to

$$\left(\frac{1 - e^{-A\mu(\Sigma)t}}{A\mu(\Sigma)t}\right)^{p/2} \leqslant \frac{e^{A\mu(\Sigma)pt} - e^{-A\mu(\Sigma)pt}}{A\mu(\Sigma)pt}.$$

But the last estimate is obvious because the quantity on the left is always less than 1, while that on the right is always greater than 1.

Thus, for $0 < p \leqslant 2$ the estimate is essentially the same as in the Logvinenko–Sereda theorem. For $p > 2$, the situation becomes quite different: we are going to construct a set Σ of finite measure such that the estimate $\|f\|^p_{L^p(\mathbb{R})} \leqslant C \int_{\mathbb{R} \setminus E} |f|^p \, d\mu$ is impossible if E is a set of positive measure. The following simple and well-known lemma will play a crucial role in our construction.

Lemma. *Let $\varphi \in S$, $A \neq 0$. Then*

$$\left| \int_{\mathbb{R}} \varphi(x) e^{2\pi i A x^2} \, dx \right| \leqslant \frac{1}{\sqrt{|A|}} \|\varphi'\|_{L^1(\mathbb{R})}.$$

Proof. Putting $v(x) \overset{\text{def}}{=} \int_x e^{2\pi i t^2} \, dt$ ($x \in \mathbb{R}$), we easily check that $|v(x)| \leqslant 1$ for every $x \in \mathbb{R}$. We assume, for definiteness, that $A > 0$. Then

$$\int_{\mathbb{R}} \varphi(x) e^{2\pi i A x^2} \, dx = \int_{\mathbb{R}} \varphi(x) v'(\sqrt{A}x) \, dx = -\frac{1}{\sqrt{A}} \int_{\mathbb{R}} \varphi'(x) v(\sqrt{A}x) \, dx.$$

It remains to observe that $\left|\int_{\mathbb{R}} \varphi'(x) v(\sqrt{A}x) \, dx\right| \leqslant \|\varphi'\|_{L^1(\mathbb{R})}$.

Corollary. *For every $\lambda, b \in \mathbb{R}$*

$$\left|\int_{\mathbb{R}} \varphi(x) e^{2\pi i(Ax^2+\lambda x+b)}\, dx\right| \leqslant \frac{1}{\sqrt{|A|}} \|\varphi'\|_{L^1(\mathbb{R})}.$$

Proof. A linear change of variable.

Now we fix $\varepsilon > 0$ and choose an arbitrary C^∞-smooth 1-periodic function g_ε satisfying $\int_0^1 g_\varepsilon\, d\mu = 1$, $g_\varepsilon \equiv 0$ on $\left(-\frac{1}{2}, \frac{1}{2}\right) \setminus \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$, and $|g_\varepsilon| \leqslant \frac{2}{\varepsilon}$. Let $J \subset \mathbb{R}$ be an interval, $f \in S$, and $\operatorname{spec} f \subset J$. We fix some numbers $A > 0$ and $b \in [0,1]$ and put $\widehat{F}(\lambda) = \widehat{f}(\lambda) g_\varepsilon(A\lambda^2 + b)$. Clearly, $F \in S$ and $\operatorname{spec} F \subset \{\, \lambda \in J : A\lambda^2 + b \in \operatorname{supp} g_\varepsilon \,\} \stackrel{\text{def}}{=} \Sigma(b)$. We have $\int_0^1 \mu\big(\Sigma(b)\big) \leqslant \varepsilon|J|$, and, therefore, we can choose b so that $\mu\big(\Sigma(b)\big) \leqslant \varepsilon|J|$. Besides, we always have $\|F\|_{L^2(\mathbb{R})} = \|\widehat{F}\|_{L^2(\mathbb{R})} \leqslant \frac{2}{\varepsilon}\|f\|_{L^2(\mathbb{R})}$. To estimate the difference $|F(x) - f(x)|$ ($x \in \mathbb{R}$), we expand g_ε in Fourier series: $g_\varepsilon(t) = \sum_{k\in\mathbb{Z}} \hat{g}_\varepsilon(k) e^{2\pi i k t}$. Our choice of g_ε immediately yields $\hat{g}_\varepsilon(0) = 1$, $\sum_{k\in\mathbb{Z}} |\hat{g}_\varepsilon(k)| < +\infty$. Therefore,

$$\begin{aligned}
|F(x) - f(x)| &= \left|\int_{\mathbb{R}} \big(\widehat{F}(\lambda) - \widehat{f}(\lambda)\big) e^{2\pi i \lambda x}\, d\lambda\right| \\
&= \left|\int_{\mathbb{R}} \Bigg(\sum_{k\in\mathbb{Z}\setminus\{0\}} \widehat{f}(\lambda)\hat{g}_\varepsilon(k) e^{2\pi i k(A\lambda^2+b)}\Bigg) e^{2\pi i\lambda x}\, d\lambda\right| \\
&\leqslant \sum_{k\in\mathbb{Z}\setminus\{0\}} \big|\hat{g}_\varepsilon(k)\big| \left|\int_{\mathbb{R}} \widehat{f}(\lambda) e^{2\pi i k(Ax^2+\lambda x+b)}\, d\lambda\right| \\
&\leqslant \sum_{k\in\mathbb{Z}\setminus\{0\}} \big|\hat{g}_\varepsilon(k)\big| \frac{1}{\sqrt{|k|A}} \|(\widehat{f})'\|_{L^1(\mathbb{R})} \leqslant \frac{1}{\sqrt{a}} \|(\widehat{f})'\|_{L^1(\mathbb{R})} \sum_{k\in\mathbb{Z}} \big|\hat{g}_\varepsilon(k)\big|.
\end{aligned}$$

For fixed ε and f, the last expression can be made arbitrarily small if we take A large enough. So, if $p > 2$, then the norm $\|f - F\|^p_{L^p(\mathbb{R})}$ is also arbitrarily small provided that A is large enough, because

$$\begin{aligned}
\|f - F\|^p_{L^p(\mathbb{R})} &\leqslant \|f - F\|^2_{L^2(\mathbb{R})} \|f - F\|^{p-2}_{L^\infty(\mathbb{R})} \\
&\leqslant \left(\frac{2}{\varepsilon} + 1\right)^2 \|f\|^2_{L^2(\mathbb{R})} \left\{\frac{1}{\sqrt{A}} \|(\widehat{f})'\|_{L^1(\mathbb{R})} \sum_{k\in\mathbb{Z}} |\hat{g}_\varepsilon(k)|\right\}^{p-2}.
\end{aligned}$$

We get the following proposition.

Proposition. *Let $p > 2$, $f \in S$, $\operatorname{spec} f \subset J$. Then for every $\mu, \delta > 0$ there exists a set $\Sigma = \Sigma(f, \mu, \delta) \subset J$ and a function $F \in S$ with spectrum contained in Σ such that $\mu(\Sigma) \leqslant \mu$ and $\|f - F\|_{L^p(\mathbb{R})} \leqslant \delta$.*

Now the construction of a "bad" set Σ is easy. Let $f_0 \in S$ be a function with spectrum contained in $J_0 \stackrel{\text{def}}{=} (1,2)$. We fix an arbitrary sequence of positive numbers $\mu_0, \mu_1, \mu_2, \dots$ satisfying $\sum_{k=0}^\infty \mu_k < +\infty$. If $f_k(x) \stackrel{\text{def}}{=} 2^{k/p} f_0(2^k x)$ ($k = 0, 1, 2, \dots$), then $\operatorname{spec} f_k \subset J_k \stackrel{\text{def}}{=}$

$(2^k, 2^{k+1})$ and $\|f_k\|_{L^p(\mathbb{R})} = \|f_0\|_{L^p(\mathbb{R})}$. Also, we fix a sequence $\delta_k \downarrow 0$. By the above proposition, we can choose $F_k \in S$ and $\Sigma_k \subset J_k$ satisfying the inequalities $\mu(\Sigma_k) \leqslant \mu_k$, $\|F_k - f_k\|_{L^p(\mathbb{R})} \leqslant \delta_k$. We put $\Sigma \stackrel{\text{def}}{=} \bigcup_{k=0}^{\infty} \Sigma_k$.

Now let $E \subset \mathbb{R}$, $\mu(E) > 0$. Assume that 0 is a density point of E. (This leads to no loss of generality because a shift does not change the spectrum.) The functions F_k satisfy

$$\|F_k\|_{L^p(\mathbb{R})} \geqslant \|f_k\|_{L^p(\mathbb{R})} - \delta_k = \|f_0\|_{L^p(\mathbb{R})} - \delta_k \to \|f_0\|_{L^p(\mathbb{R})},$$

$$\left(\int_{\mathbb{R}\setminus E} |F_k|^p \, d\mu\right)^{1/p} \leqslant \delta_k + \left(\int_{\mathbb{R}\setminus E} |f_k|^p \, d\mu\right)^{1/p} = \delta_k + \left(\int_{\mathbb{R}\setminus 2^k E} |f_0|^p \, d\mu\right)^{1/p} \to 0.$$

So, certain functions with spectrum contained in Σ may be almost supported on E in the L^p-sense.

Finally, we note that the set Σ constructed above is very simple from the topological point of view: the intersection of E with every bounded interval is merely a union of a finite number of disjoint closed intervals (therefore, it does not matter how one understands the inclusion $\operatorname{spec} f \subset \Sigma$). Besides, it follows from our construction that Σ may thin out arbitrarily fast at infinity, i.e., it can be ensured that the function $t \to \mu\big(\Sigma \setminus [-t, t]\big)$ be majorized by a prescribed positive monotone decreasing function.

§4.3. The Mandelbrojt theorem

In this section we discuss the sharpness of Theorem 3.6 from §4 of Chapter 3. We recall a well-known theorem of Mandelbrojt on the behavior of functions with "thin" spectrum on small arcs.

Theorem. *Let* $0 < \sigma < 1$, $\Lambda \subset \mathbb{Z}$. The following statements are equivalent:

(a) Among every N successive integers there is at most $\text{const} N^\sigma$ elements of Λ;

(b) For every function $f \in L^2(\mathbb{T})$ with spectrum contained in Λ and every $\mathfrak{L} \subset \mathbb{T}$

$$\int_{\mathfrak{L}} |f|^2 \, d\mu \leqslant \text{const} |\mathfrak{L}|^{1-\sigma} \|f\|^2_{L^2(\mathbb{T})};$$

(c) For every function $f \in L^2(\mathbb{T})$ with spectrum contained in Λ and every $\mathfrak{L} \subset \mathbb{T}$

$$\|f\|^2_{L^2(\mathbb{T})} \leqslant \exp\big\{\text{const} |\mathfrak{L}|^{-\frac{\sigma}{1-\sigma}}\big\} \int_{\mathfrak{L}} |f|^2 \, d\mu.$$

Clearly, this theorem implies that to prove the sharpness of Theorem 3.6 (stated in §4 of Chapter 3) it suffices to construct a set $\Lambda \subset \mathbb{Z}$ satisfying the Zygmund condition and such that $\operatorname{card} \Lambda \cap [0, \mathbb{N}] \geqslant c N^{\frac{1}{2}-\varepsilon}$ for each $N = 1, 2, \dots$. Simple computations show that as Λ we can take the random set $\{\, n \in \mathbb{N} : \xi_n = 1 \,\}$, where ξ_n are independent random variables defined by $P(\xi_n = 1) = n^{-\frac{1}{2}-\varepsilon}$, $P(\xi_n = 0) = 1 - n^{-\frac{1}{2}-\varepsilon}$.

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