# On the behavior of Lebesgue constants for two-dimensional Fourier sums over polygons

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## §0. Introduction.

For every function $f \in L^1_{\text{loc}}(\mathbb{R}^2)$ which is 1-periodic with respect to each variable one can define its Fourier series by $f(t,s) \sim \sum\limits_{m,n\in\mathbb{Z}} \widehat{f}(m,n)\, e^{2\pi i(mt+ns)}$ where $\widehat{f}(m,n) = \iint\limits_{[-1/2,1/2]^2} f(t,s)\; e^{-2\pi i(mt+ns)}\; dt\; ds$. Unlike the case of one variable, there is no canonical way to summarize this series. One can choose an arbitrary (closed) bounded domain $W \subset \mathbb{R}^2$ (containing the origin inside it) and put

$$S_\lambda\; f(t,s) = \sum_{(m,n)\in\lambda W} \widehat{f}(m,n)\; e^{2\pi i(mt+ns)}.$$

This "partial sum" can also be written as a convolution $S_\lambda\; f = f * \mathcal{D}_\lambda$ where $\mathcal{D}_\lambda(t,s) = \sum\limits_{(m,n)\in\lambda W} e^{2\pi i(mt+ns)}$ is the corresponding Dirichlet's kernel.

The choice of the concrete domain $W$ influences the behavior of the sequence of partial sums $S_\lambda\; f(t,s)$ very much. We shall restrict ourselves to the case when $W$ is a closed polygon (of very general nature: we assume it neither convex, nor connected; just bounded). This choice is the best possible one in many respects. For example, as it was shown by Fefferman, the Carleson-Hunt theorem on the convergence almost everywhere remains valid (and moreover, is a simple consequence of its one-dimensional version). Our aim is to investigate the growth of the so-called Lebesgue constants $L_\lambda \overset{\text{def}}{=} \iint\limits_{[-1/2,1/2]^2} |\mathcal{D}_\lambda(t,s)|\; dt\; ds$ when $\lambda \to +\infty$. Let's recall what has been already known about them

1) for every polygon $W$ $\quad c(W)\log^2\lambda \le L_\lambda \le C(W)\log^2\lambda$ for large $\lambda$ $(0 < c(W) < C(W) < +\infty)$

2) if slopes of all sides of $W$ are rational, $\mathcal{D}_\lambda(t,s)$ has almost the same behavior as its continuous analogue $\widetilde{\mathcal{D}}_\lambda(t,s) = \iint\limits_{\lambda W} e^{2\pi i(mt+ns)}\, dm\, dn$; namely, $\iint\limits_{[-1/2,1/2]^2} |\mathcal{D}_\lambda - \widetilde{\mathcal{D}}_\lambda|\, dt\, ds \le C(W)\log\lambda$. The kernel $\widetilde{\mathcal{D}}_\lambda$ is much more handable than $\mathcal{D}_\lambda$ in general case and for reasonable domains $W$ the integral of its absolute value over $[-1/2,1/2]^2$ has a good asymptotics not very difficult to calculate. In the case of $W$ being a polygon $\iint\limits_{[-1/2,1/2]^2} |\widetilde{\mathcal{D}}_\lambda|\, dt\, ds \sim c(W)\log^2\lambda$. The constant $c(W)$ can be easily computed when $W$ has no parallel sides: it equals $\frac{2}{\pi^3} \times \{\text{number of sides of } W\}$. In general case the explicit expression for $c(W)$ is more complicated and we refer the reader to [] for details.

3) In [] the case of $W$ being a triangle $W_\alpha = \{(u,v) : 0 \le v \le \alpha u \le \alpha\}$ was investigated. It was shown that

$$L_\lambda = \iint\limits_{[-1/2,1/2]^2} |\widetilde{\mathcal{D}}_\lambda|\, dt\, ds + \int_0^1 \left| \sum_{v=0}^{\lambda} \{\alpha v\}\, e^{2\pi i v t} \right| dt + O(\log\lambda \log\log\lambda) \qquad (*)$$

for large $\lambda$ and that there exists $\alpha > 0$ for wich the second term is greater than $\frac{1}{4\pi^2}\log^2\lambda$ along some (very sparse) sequence of $\lambda$ tending to $+\infty$ (note that the first term is equivalent to $\frac{6}{\pi^3}\log^2\lambda$). It was shown also that one can choose $\alpha$ in such a way that the ratio $\frac{L_\lambda}{\log^2\lambda}$ has no limit when $\lambda \to +\infty$. The main drawback of the results in [] was that they concerned only the values of $\alpha$ belonging to a very thin set and gave no information about other $\alpha \in (0,+\infty)$ at all. In this paper this uncertainty will be partly overcome: we shall prove that $L_\lambda$ is always essentially greater than $\iint\limits_{[-1/2,1/2]^2} |\widetilde{\mathcal{D}}_\lambda|\, dt\, ds$ unless all the slopes of sides of $W$ are rational, but the question whether the limit $\lim\limits_{\lambda\to\infty} \frac{L_\lambda}{\log^2\lambda}$ may exist for some polygon $W$ with irrational slopes of sides, remains open. As the authors have different opinions of it, we give no conjecture here.

For reader's convenience we shall deal with the triangle $W_\alpha$ instead of an arbitrary polygon $W$ up to the last paragraph, where we shall describe the changes in proofs to be made and the additional ideas to be involved in general case. It will allow us to avoid clumsy formulas and computations and make the main idea of the proof clear and easy to understand.

§1. **Preliminary results.**

Here we shall briefly recall the derivation of $(*)$ in []. The reader who is familiar with it may go to §2 immediately.

We have

$$\begin{aligned}\mathcal{D}_\lambda(t,s) &= \sum_{0\le m\le\lambda}\ \sum_{0\le n\le\alpha m} e^{2\pi i(mt+ns)} = \sum_{0\le m\le\lambda} e^{2\pi imt}\int_{[0,\alpha,n]} e^{2\pi ins}\, d[n] \\ &= \sum_{0\le m\le\lambda} e^{2\pi imt}\int_{[0,\alpha m]} e^{2\pi ins}\, dn - \sum_{0\le m\le\lambda} e^{2\pi imt}\int_{[0,\alpha n]} e^{2\pi ins}\, d\{n\} \\ &\stackrel{\text{def}}{=} S_1(t,s) - S_2(t,s).\end{aligned}$$

Note that $S_1(t,s) = \frac{1}{2\pi is}\sum\limits_{0\le m\le\lambda}\left(e^{2\pi im(t+\alpha s)} - e^{2\pi imt}\right)$ while

$$\widetilde{\mathcal{D}}_\lambda(t,s) = \frac{1}{2\pi is}\int_0^\lambda \left(e^{2\pi im(t+\alpha s)} - e^{2\pi imt}\right)\, dm.$$

Beside it, obviously $|S_1(t,s)| + |\widetilde{\mathcal{D}}_\lambda(t,s)vert \le C\lambda^2$ for large $\lambda$. Thus

$$\begin{aligned}\iint\limits_{[-1/2,1/2]^2} |S_1 - \widetilde{\mathcal{D}}_\lambda|\, dt\, ds &= \int_{|s|\le\frac{1}{\lambda^2}}\int_{|t|\le 1/2} + \int_{1/\lambda^2\le|s|\le\frac{1}{4\alpha}}\int_{|t|\le1/2} + \int_{|s|\ge\frac{1}{4\alpha}}\int_{|t|\le1/2} \\ &\stackrel{\text{def}}{=} I_1 + I_2 + I_3,\end{aligned}$$

where

$$|I_1| \le \int_{|s|\le 1/\lambda^2}\int_{|t|\le 1/2} C\lambda^2\, dt\, ds \le \text{Const},$$

$$|I_3| \le 4\alpha \iint\limits_{[-1/2,1/2]^2} |s|\,\left(|S_1(t,s)| + |\widetilde{\mathcal{D}}(t,s)|\right)\, dt\, ds \le C\log\lambda$$

(because we integrate the sum of 4 one-dimensional Dirichlet's kernels).

At last, to estimate $I_2$, note that

$$\left|\sum_{0\le m\le\lambda} e^{2\pi im\xi} - \int_0^\lambda e^{2\pi im\xi}\, dm\right| \le \text{Const} \quad \text{for all} \quad \xi\in(-3/4,3/4)$$

and therefore

$$|I_2| \le \int_{1/\lambda^2\le|s|\le1/2} \frac{\text{Const}}{|s|} \le C\log\lambda$$

as well and we get

$$\iint\limits_{[-1/2,1/2]^2} \left| S_1(t,s) - \widetilde{\mathcal{D}}_\lambda(t,s) \right| \, dt \, ds = O(\log \lambda) \qquad (\lambda \to +\infty).$$

From other side, integrating by parts, we obtain:

$$S_2(t,s) = \sum_{0 \le m \le \lambda} e^{2\pi i m t} + \sum_{0 \le m \le \lambda} \{\alpha m\} \, e^{2\pi i m (t+\alpha s)} - \sum_{0 \le m \le \lambda} e^{2\pi i m t} \int_0^{\alpha m} 2\pi i s \, \{n\} \, e^{2\pi i s n} \, dn.$$

The first term is just a one-dimensional (one-sided) Dirichlet's kernel $D_\lambda(t)$ and its $L^1$-norm doesn't exceed $C \log \lambda$.

To estimate the $L^1$-norm of the last term, it is convenient to decompose $\{n\}$ into its Fourier series: $\{n\} = \frac{1}{2} + \sum\limits_{r=0} \frac{1}{2\pi i r} \, e^{2\pi i r n}$. We get

$$\int_0^{\alpha_m} \{n\} \, e^{2\pi i s n} \, dn = \frac{1}{4\pi i s} \left( e^{2\pi i m \alpha s} - 1 \right) + \sum_{r \ne 0} \frac{1}{(2\pi i r) \, 2\pi i (r+s)} \left( e^{2\pi i m \alpha (r+s)} - 1 \right)$$

and therefore the last term is equal to

$$\frac{1}{2} \Big( D_\lambda(t+\alpha s) - D_\lambda(t) \Big) + \sum_{r \ne 0} \frac{s}{2\pi i r (r+s)} \Big( D_\lambda(t + \alpha(r+s)) - D_\lambda(t) \Big).$$

It is easy to derive from this representation that the $L^1$-norm to be estimated is less than $C \log \lambda$ as well.

Combining all these results, we obtain

$$\mathcal{D}_\lambda(t,s) = \widetilde{\mathcal{D}}_\lambda(t,s) - \sum_{0 \le m \le \lambda} \{\alpha m\} \, e^{2\pi i m (t+\alpha s)} + R(t,s)$$

with $\iint\limits_{[-1/2,1/2]^2} |R(t,s)| \, dt \, ds \le C \log \lambda$.

To derive $(*)$ from here one is just to note that $\widetilde{\mathcal{D}}(t,s)$ is in essential concentrated at small values of $s$ while the sum $\sum\limits_{0 \le m \le \lambda} \{\alpha m\} \, e^{2\pi i m (t+\alpha s)}$ is obviously "equidistributed" on $[-1/2, 1/2]$ with respect to $s$ in $L^1$-sense. More precisely,

$$\int_{\frac{1}{\log \lambda} \le |s| \le 1/2} ds \left( \int_{|t| \le 1/2} dt |\widetilde{\mathcal{D}}(t,s)| \right) \le \int_{\frac{1}{\log \lambda} \le |s| \le 1/2} \frac{C \log \lambda}{|s|} = O(\log \lambda \, \log \log \lambda)$$

while

$$\int_{|s| \le \frac{1}{\log \lambda}} ds \left( \int_{|t| \le 1/2} dt \left| \sum_{0 \le m \le \lambda} \{\alpha m\} \, e^{2\pi i m (t+\alpha s)} \right| \right)$$

$$= \frac{2}{\log \lambda} \int_0^1 \left| \sum_{0 \le v \le \lambda} \{\alpha v\} \, e^{2\pi i v t} \right| dt = O(\log \lambda)$$

(the last inequality is well-known and we shall prove it below by pass).

§2. **The formulation of the main proposition.**

The most part of the information about the integral $I_\lambda(\alpha) \stackrel{\text{def}}{=} \int_0^1 \Big| \sum\limits_{0\le v\le\lambda} \{\alpha v\}\, e^{2\pi i v t}\Big| dt$ we possess is contained in the following

**Proposition.**

1) if $\alpha = \frac{p}{q} \in \mathbb{Q}$ then $\overline{\lim\limits_{\lambda\to\infty}}\ \frac{I_\lambda(\alpha)}{\log\lambda} \le \sup\limits_{\lambda\ge 2} \frac{I_\lambda(\alpha)}{\log\lambda} \le C\log q$;

2) if $\alpha \notin \mathbb{Q}$ then $c \le \overline{\lim\limits_{\lambda\to\infty}}\ \frac{I_\lambda(\alpha)}{\log^2\lambda} \le C$, constants $c$ and $C$ being absolute;

3) $\underline{\lim\limits_{\lambda\to\infty}}\ \frac{I_\lambda(\alpha)}{\log^2\lambda} > 0$ iff $\alpha$ is not a Liouvillian number, i.e. iff there exists $M > 0$ such that $|\alpha - \frac{p}{q}| \ge \frac{\text{const}}{q^M}$ for all $p, q \in \mathbb{N}$;

4) if $\alpha \notin \mathbb{Q}$ and $|\alpha - \frac{p}{q}| \le \frac{1}{q^M}$ with some $M > 2$ for infinitely many rationals $\frac{p}{q}$, then no limit $\lim\limits_{\lambda\to\infty} \frac{I_\lambda(\alpha)}{\log^2\lambda}$ exists;

5) the integral $I_\lambda(\alpha)$ is concentrated at a set of small measure, namely, for each $\lambda \ge 2$ and $\alpha \notin \mathbb{Q}$ there exists a set $E = E(\lambda,\alpha) \subset (0,1)$ such that $\mu(E) \le e^{-\sqrt{\log\lambda}}$ while $\int_{[0,1]\setminus E} \Big| \sum\limits_{0\le v\le\lambda} \{\alpha v\}\, e^{2\pi i v t}\Big|\, dt \le C\log^{3/2}\ \lambda$.

It follows from (2) & (3) that the "typical" behavior of $I_\lambda(\alpha)$ from the viewpoint of category is that $\underline{\lim\limits_{\lambda\to\infty}}\ \frac{I_\lambda(\alpha)}{\log^2\lambda} = 0$ while $\overline{\lim\limits_{\lambda\to\infty}}\ \frac{I_\lambda(\alpha)}{\log^2\lambda} \ge c > 0$ (just because the set of non-Liouvillian numbers is of the first category in $(0,+\infty)$).

It is also not very difficult to show that the typical behavior from the viewpoint of Lebesgue measure exists as well. As

$$\sum_{0\le v\le\lambda/2} \{2\alpha v\}\, e^{2\pi i(2v)t} = \frac{1}{2}\left( \sum_{0\le v\le\lambda} \{\alpha v\}\, e^{2\pi i v t} + \sum_{0\le v\le\lambda} \{\alpha v\}\, e^{2\pi i v(t+\frac{1}{2})} \right),$$

we get $I_{\lambda/2}(\{2\alpha\}) \le I_\lambda(\alpha)$. If we denote

$$\omega(\alpha) \stackrel{\text{def}}{=} \underline{\lim_{\lambda\to\infty}}\ \frac{I_\lambda(\alpha)}{\log^2\lambda} \qquad \text{and} \qquad \Omega(\alpha) \stackrel{\text{def}}{=} \overline{\lim_{\lambda\to\infty}}\ \frac{I_\lambda(\alpha)}{\log^2\lambda},$$

we obtain immediately from here that $\omega(\{2\alpha\}) \le \omega(\alpha)$ and $\Omega(\{2\alpha\}) \le \Omega(\alpha)$ which (together with the ergodicity of the transform $[0,1) \ni \alpha \mapsto \{2\alpha\} \in [0,1)$) implies $\omega(\alpha) = \omega$, $\Omega(\alpha) = \Omega$ for almost all $\alpha \in [0,1)$ with some $0 < \omega \le \Omega < +\infty$. Very interesting unsolved question is whether $\omega = \Omega$ or not.

At last it is worth mentioning that the statements (2)-(4) have their local versions. We shall prove below that for any interval $\Delta \subset \mathbb{R}$ the difference of the integrals of $\Big| \sum\limits_{0\le v\le\lambda} \{\alpha v\}\ e^{2\pi i v t}\Big|$ over $\Delta$ and $\Delta+\alpha$ doesn't exceed $C \log\lambda$. This (together with (2) and the well-known fact that the numbers $\{\alpha\}$, $\{2\alpha\}$, $\{3\alpha\}$,... are equidistributed on $(0,1)$ when $\alpha \notin \mathbb{Q}$) implies

$$\int_\Delta \Big| \sum_{0\le v\le\lambda} \{\alpha v\}\ e^{2\pi i v t}\Big|\ dt = \mu(\Delta)\ I_\lambda(\alpha) + o(\log^2\lambda)$$

where the explicit upper estimate for $o(\log^2\lambda)$ depends on $\alpha$ and $\mu(\Delta)$ only (not on the situation of the interval $\Delta$ on the real line). For $\mathcal{D}_\lambda(t,s)$ it means that each point $(t,s) \neq (0,0)$ gives approximately the same contribution into $L_\lambda$ and therefore no "localization principle" exists (unlike the case of a rational slope, when it is easy to show that the sequence of partial sums $S_\lambda\ f$ of the function $f$ supported in $\varepsilon$-neighbourhood of the origin converges to 0 outside several strips of width $2\varepsilon$ and even inside these strips $S_\lambda(t,s)$ can grow only as $\log\lambda$ (not as $L_\lambda \sim \log^2\lambda$) unless $(t,s) \in \operatorname{supp} f$.)

Let's now go to the proofs. We may obviously suppose that $\alpha \in (0,1)$ and $\lambda \in \mathbb{N}$.

## §3. Estimates of $I_\lambda(\alpha)$ from above.

First of all consider the case $\alpha = \frac{p}{q}$ with $(p,q) = 1$. Define 1-periodic function $\varphi(s)$ by

$$\varphi(s) = \begin{cases} s & , \quad 0 \le s \le \frac{q-1}{q} \\ (q-1)(1-s) & , \quad \frac{q-1}{q} \le s \le 1. \end{cases}$$

The Fourier coefficients of $\varphi$ are $\widehat{\varphi}(0) = \frac{q-1}{2q}$ and $\widehat{\varphi}(k) = \frac{q(1-e^{-2\pi i k/q})}{(2\pi i k)^2}$ when $k \in \mathbb{Z}$, $k \ne 0$. It is easy to derive from here that

$$\frac{1}{\pi^2 |k|} \le |\widehat{\varphi}(k)| \le \frac{1}{2\pi |k|} \quad \text{for} \quad 0 < |k| \le q/2 \quad \text{and} \quad |\widehat{\varphi}(k)| \le \frac{q}{2\pi^2 k^2}$$

for all $k \in \mathbb{Z}$.

Note that

$$\begin{aligned} \sum_{0 \le v \le \lambda} \{\alpha v\}\, e^{2\pi i v t} &= \sum_{0 \le v \le \lambda} \varphi(\alpha v)\, e^{2\pi i v t} = \sum_{0 \le v \le \lambda} \left( \sum_{k \in \mathbb{Z}} \widehat{\varphi}(k)\, e^{2\pi i k \alpha v} \right) e^{2\pi i v t} \\ &= \sum_{k \in \mathbb{Z}} \widehat{\varphi}(k) \sum_{0 \le v \le \lambda} e^{2\pi i v (t + k\alpha)} = \sum_{k \in \mathbb{Z}} \widehat{\varphi}(k)\, D_\lambda(t + k\alpha) \end{aligned}$$

(we denote by $D_\lambda$ one-dimensional Dirichlet's kernel $D_\lambda(t) = \sum\limits_{0 \le v \le \lambda} e^{2\pi i v t}$ again).

Thus,

$$I_\lambda(\alpha) \le \left( \sum_{k \in \mathbb{Z}} |\widehat{\varphi}(k)| \right) \int_0^1 |D_\lambda(t)|\, dt \le C \ \log q \ \log \lambda,$$

proving (1).

For any $\alpha \in (0,1)$, $\lambda \ge 2$ one can, according to the well-known Dirichlet theorem, find some rational number $\frac{p}{q}$ such that $q \le \lambda$ and $|\alpha - \frac{p}{q}| \le \frac{1}{q\lambda}$. Obviously, we may also suppose that $(p,q) = 1$.

If $\gamma \overset{\text{def}}{=} \alpha - \frac{p}{q}$ is positive we have $\{\alpha v\} = \{\frac{p}{q}\, v\} + \gamma v$ for all $0 \le v \le \lambda$ and therefore

$$I_\lambda(\alpha) \le I_\lambda(\frac{p}{q}) + \int_0^1 \left| \sum_{0 \le v \le \lambda} \gamma v\, e^{2\pi i v t} \right| dt \le C \log q \log \lambda + C\gamma\lambda \log \lambda \le C \log^2 \lambda + C \log \lambda$$

for $\gamma\lambda \le 1/q \le 1$.

If $\gamma$ is negative, the equality $\{\alpha v\} = \{\frac{p}{q}\, v\} + \gamma v$ still holds, except the values of $v$ divisible by $q$ at which the left and the right part differ by 1. This gives an additional term $\int_0^1 |D_{\lambda/q}(qt)|\, dt$ which is less than $C \log \lambda$ as well, proving a half of (2).

Note also that the sum $\sum\limits_{0\le v\le\lambda} \widehat{\varphi}(k)\ D_\lambda(t+k\ \frac{p}{q})$ can be considered as a partial sum of the Fourier series of the (complex-valued) measure $\nu \stackrel{\text{def}}{=} \sum\limits_{k\in\mathbb{Z}} \widehat{\varphi}(k)\ \delta_{\{-k\ \frac{p}{q}\}}$ and therefore satisfies the weak-type estimate

$$\mu\left\{t\in(0,1):\left|\sum_{k\in\mathbb{Z}}\widehat{\varphi}(k)\ D_\lambda\left(t+\frac{kp}{q}\right)\right|\ge y\right\}\le\frac{C\|\nu\|}{y}\le\frac{C\log q}{y}\le\frac{C\log\lambda}{y}$$

for all $y>0$.

As this sum approximates the sum $\sum\limits_{0\le v\le\lambda}\{\alpha v\}\ e^{2\pi i v t}$ up to the term of $L^1$-norm $C\log\lambda$ or less, the same weak type etimates holds for $\sum\limits_{0\le v\le\lambda}\{\alpha v\}\ e^{2\pi i v t}$. So we may put

$$E(\lambda,\alpha)\ \stackrel{\text{def}}{=}\ \left\{t\in(0,1):\left|\sum_{0\le v\le\lambda}\{\alpha v\}\ e^{2\pi i v t}\right|\ge C\log\lambda\ e^{\sqrt{\log\lambda}}\right\},$$

proving (5).

Suppose at last that $|\alpha-\frac{p}{q}|\le\frac{1}{q^M}$ with large $q$ and $M$. Putting $\lambda=q^{M-1}$, we get $I_\lambda(\alpha)\le C\log q\ \log\lambda+C\log\lambda=C\big(\frac{1}{M-1}+\frac{1}{\log\lambda}\big)\ \log^2\lambda$ which is small in comparison with $\log^2\lambda$. Thus, the "only if" part of (3) is also proved.

As for the remark about the difference between the integrals over $\Delta$ and $\Delta+\alpha$, it follows immediately from the representation

$$\sum_{0\le v\le\lambda}\{\alpha v\}\ e^{2\pi i v t}-\sum_{0\le v\le\lambda}\{\alpha v\}\ e^{2\pi i v(t+\alpha)}=\sum_{k\in\mathbb{Z}}\widehat{\psi}(k)\ D_\lambda(t+\alpha k)+\{\alpha(\lambda+1)\}\ e^{2\pi i(\lambda+1)(t+\alpha)},$$

where $\psi(t)$ is the 1-periodic continuation of the function $s(e^{2\pi i s}-1)$, if we take into account that the Fourier series of $\psi$ is asolutely convergent.

## §4. **The estimate of $I_\lambda(\alpha)$ from below.**

We shall start with the following simple

**Lemma.**

Let $\left|\alpha-\frac{P}{Q}\right|\le\frac{1}{Q^2}$ and $(P,Q)=1$. Then for each $\lambda\in\left(Q^4,e^{\sqrt{Q}}\right)$ the inequality $I_\lambda\ge c\log Q\log\lambda$ holds with some absolute constant $c>0$.

**Proof.**

Without loss of generality we may suppose that $Q$ is not very small (at any rate we are going to apply this lemma to $Q\to+\infty$).

As it was shown in §3, we may approximate $\sum\limits_{0\le v\le\lambda}\{\alpha v\}\,e^{2\pi ivt}$ by $\sum\limits_{k\in\mathbb{Z}}\widehat{\varphi}(k)\,D_\lambda(t+k\,\frac{p}{q})$ for some rational $\frac{p}{q}$ satisfying $|\alpha-\frac{p}{q}|\le\frac{1}{q\lambda}$ and $q\le\lambda$ with function $\varphi$ corresponding to this denominator $q$. In the next paragraph it will be very convenient that in this expression $D_\lambda(t+k\,\frac{p}{q})$ stands here instead of (more natural) term $D_\lambda(t+k\alpha)$, but now it isn't so. Nevertheless, as $\int_0^1|\frac{d}{dt}\,D_\lambda(t)|\,dt\le C\lambda\log\lambda$, we have

$$\sum_{k\in\mathbb{Z}}|\widehat{\varphi}(k)|\int_0^1\left|D_\lambda(t+k\,\frac{p}{q})-D_\lambda(t+k\alpha)\right|$$
$$\le\sum_{|k|\le q/2}|\widehat{\varphi}(k)|\;|k|\;|\alpha-\frac{p}{q}|\int_0^1\left|\frac{d}{dt}\,D_\lambda(t)\right|\,dt+2\sum_{|k|>q/2}|\widehat{\varphi}(k)|\int_0^1|D_\lambda(t)|\,dt$$
$$\le\sum_{0<|k|\le q/2}\frac{1}{2\pi|k|}\;|k|\;\frac{1}{q\lambda}\;C\lambda\log\lambda+2\sum_{k>q/2}\frac{q}{2\pi^2k^2}\;C\log\lambda\le C\log\lambda,$$

and therefore the approximation by the sum $\sum\limits_{k\in\mathbb{Z}}\widehat{\varphi}(k)\;D_\lambda(t+k\alpha)$ gives an error term of $L^1$-norm $C\log\lambda$ or less as well.

Fix some $k_0\in\left(0,\sqrt[3]{Q}\right)$ and consider the interval $\Delta=\Delta(k_0)$ of length $\frac{1}{Q^2}$ centered at $-k_0\alpha$. Note that for each integer $\ell$ satisfying $|\ell|\le\frac{Q}{2}$ the inequality $\left|\ell\alpha-\frac{\ell P}{Q}\right|\le\frac{|\ell|}{Q^2}\le\frac{1}{2Q}$ holds and therefore the distance from $\ell\alpha$ to the nearest integer is not less than $\frac{1}{Q}-\frac{1}{2Q}=\frac{1}{2Q}$. This, together with the obvious estimate $|D_\lambda(\xi)|=\left|\frac{e^{2\pi i\lambda\xi}-1}{e^{2\pi i\xi}-1}\right|\le\frac{1}{2|\xi|}$ for $|\xi|\le 1/2$, implies that $|D_\lambda(t+k\alpha)|\le\frac{1}{1/Q-1/Q^2}\le 2Q$ for all $t\in\Delta$ unless $k$ belongs to some exceptional set $\Lambda\ni k_0$ satisfying $|k'-k''|>\frac{Q}{2}$ for each two different elements $k',k''\in\Lambda$. Beside it, we get also from here that the intervals $\Delta(k)$, $k=1,2,\ldots,\left[\sqrt[3]{Q}\right]$, are mutually disjoint (in 1-periodic sense).

We have

$$\int_\Delta \left| \sum_{k\in\mathbb{Z}} \widehat{\varphi}(k)\ D_\lambda(t+k\alpha)\right| \, dt \geq |\widehat{\varphi}(k_0)| \int_\Delta |D_\lambda(t+k_0\alpha)| \, dt$$

$$-\int_\Delta \left| \sum_{k\in\Lambda\setminus\{k_0\}} \widehat{\varphi}(k)\ D_\lambda(t+k\alpha)\right| \, dt - \int_\Delta \left| \sum_{k\in\mathbb{Z}\setminus\Lambda} \widehat{\varphi}(k)\ D_\lambda(t+k\alpha)\right| \, dt \stackrel{\text{def}}{=} I_1 - I_2 - I_3.$$

As $\lambda \geq Q^4$,

$$\int_\Delta \left| D_\lambda(t+k_0\alpha)\right| \, dt = \int_{|t|\leq \frac{1}{2Q^2}} |D_\lambda(t)| \, dt \geq c\log\lambda$$

and therefore $I_1 \geq \frac{c\log\lambda}{\pi^2 k_0}$.

Then, it follows from the above remark immediately that

$$I_3 \leq \mu(\Delta)\ 2Q \sum_{k\in\mathbb{Z}} |\widehat{\varphi}(k)| \leq \frac{C\log q}{Q} \leq \frac{C\log\lambda}{Q}.$$

At last,

$$I_2 \leq \sum_{k\in\Lambda\setminus\{k_0\}} |\widehat{\varphi}(k)| \int_0^1 |D_\lambda(t)| \, dt \leq C\log\lambda \sum_{k\in\Lambda\setminus\{k_0\}} |\widehat{\varphi}(k)|.$$

Let's note that $|k| \geq |k-k_0| - |k_0| \geq \frac{Q}{2} - \sqrt[3]{Q} > \frac{Q}{3}$ for each $k \in \Lambda\setminus\{k_0\}$. Thus for some integer $d \geq \frac{Q}{3}$ elements of $\Lambda\setminus\{k_0\}$ satisfy $|k| \geq d$ and $|k'-k''| \geq d$ when $k' \neq k''$. Recall that $\left|\widehat{\varphi}(k)\right|$ has a symmetric and decreasing in both sides (with respect to $k$) majorant

$$\phi(k) = \begin{cases} 1/2 & , \quad k=0 \\ \frac{1}{2\pi|k|} & , \quad 0<|k|\leq \frac{q}{2} \\ \frac{q}{2\pi^2 k^2} & , \quad |k|>q/2 \end{cases}$$

and thereby

$$\sum_{k\in\Lambda\setminus\{k_0\}} |\widehat{\varphi}(k)| \leq \sum_{k\in\Lambda\setminus\{k_0\}} \phi(k) \leq 2\sum_{\ell=1}^{\infty} \phi(\ell d) \leq 2\ \frac{1}{d}\sum_{k=1}^{\infty}\phi(k) \leq \frac{C\log q}{d} \leq \frac{3C\log\lambda}{Q}.$$

So, we finally have

$$I_2 \leq \frac{C\log^2\lambda}{Q} \leq \frac{C\log\lambda}{\sqrt{Q}} \qquad (\text{for } \lambda \leq e^{\sqrt{Q}})$$

and

$$\int_{\Delta}\left|\sum_{k\in\mathbb{Z}}\widehat{\varphi}(k)\ D_\lambda(t+k\alpha)\right|\,dt \geq \left(\frac{c}{k_0}-\frac{C}{Q}-\frac{C}{\sqrt{Q}}\right)\ \log\lambda \geq \frac{c}{2k_0}\ \log\lambda$$

with some absolute constants $c$ and $C$ when $Q$ is not very small (we have used here the estimate $k_0 \leq \sqrt[3]{Q}$ which provides that the first term is actually much greater than both the terms we subtract).

Now

$$\int_0^1 \ldots \geq \sum_{1\leq k\leq\sqrt[3]{Q}}\ \int_{\Delta(k)} \ldots \geq \sum_{1\leq k\leq\sqrt[3]{Q}} \frac{C}{2k}\ \log\lambda \geq c'\log Q\log\lambda$$

and it remains only to note that subtraction of $C\log\lambda$ from the right side part (which is needed to return to the initial integral $I_\lambda(\alpha)$) doesn't influence this estimate for large $Q$. The lemma is proved.

For every $\alpha\notin\mathbb{Q}$ one can find infinitely many rationals $\frac{P}{Q}$ satisfying $|\alpha-\frac{P}{Q}|\leq\frac{1}{Q^2}$. Putting for each such $Q$ $\lambda \overset{\text{def}}{=} Q^4$, we get the lower estimate in (2) immediately.

Suppose now that $|\alpha-\frac{P}{Q}|\geq\frac{\text{const}}{Q^M}$ for all rational $\frac{P}{Q}$. For each $\lambda\geq 1$ one can find some rational $\frac{P}{Q}$ satisfying $Q\leq\sqrt[4]{\lambda}$, $(P,Q)=1$ and $|\alpha-\frac{P}{Q}|\leq\frac{1}{Q\sqrt[4]{\lambda}}\ \left(\leq\frac{1}{Q^2}\right)$. For this $Q$ we must have $\frac{1}{Q\sqrt[4]{\lambda}}\geq\frac{\text{const}}{Q^M}$ and therefore $Q\geq(\text{const}\,\lambda)^{\frac{1}{4(M-1)}}\geq\lambda^{\frac{1}{4M}}\geq\log^2\lambda$ for large $\lambda$. Thus all conditions of lemma are satisfied and we get

$$I_\lambda(\alpha)\geq c\log Q\log\lambda\geq\frac{c}{4M}\ \log^2\lambda$$

for large $\lambda$, proving the remaining "if" part of (3). Note that both upper and lower estimates for $\varliminf\limits_{\lambda\to\infty}\ \frac{I_\lambda(\alpha)}{\log^2\lambda}$ are of the kind $\frac{\text{const}}{M}$, i.e. this lower limit is inversely proportional to the "Liouvillian exponent" of the number $\alpha$ in essential.

§5. **Proof of (4).**

First of all note that (4) is a consequence of (2) and the quantitative version of (3) proved above when $M$ is sufficiently large and that the set of $\alpha \in (0,1)$, which can be approximated by rational numbers $\frac{p}{q}$ with an error term essentially less than $1/q^2$, has zero measure. Thus, the result we are going to prove here is not more than a curious observation. Nevertheless, why not to prove it if we are able to do that?

Suppose the contrary, i.e. that the limit $L(\alpha) = \lim\limits_{\lambda\to\infty} \frac{I_\lambda(\alpha)}{\log^2\lambda}$ exists. Let $q$ be a (very large) denominator for which $|\alpha - \frac{p}{q}| \le \frac{1}{q^M}$. It has been shown in §2 that $|I_\lambda(\alpha) - I_\lambda(\frac{p}{q})| \le C\log\lambda$ for all $0 \le \lambda \le q^{M-1}$. We shall consider the values of $\lambda$ for which $\lambda+1$ is divisible by $q$ (i.e. $\lambda = \ell q - 1$, $1 \le \ell \le q^{M-2}$). The existence of the limit $L(\alpha)$ implies in particular that $I_{\ell q-1}(\alpha) = L(\alpha)(\log\ell + \log q)^2 + o(\log^2 q)$ when $q \to +\infty$ ($M$ is fixed!). And we shall show that

$$I_{\ell q-1}(\frac{p}{q}) = \frac{4}{\pi^3}\ \log q\ \log\ell + \mathrm{const}(q) + o(\log^2 q),$$

i.e. is a linear function in $\log\ell$ in essential. As the interval of possible values of $\log\ell$ has length compatible with $\log q$, this gives immediately the contradiction desired.

Write again

$$\sum_{0\le v\le \ell q-1} \left\{\frac{p}{q}\ v\right\}\ e^{2\pi i v t} = \sum_{k\in\mathbb{Z}} \widehat{\varphi}(k)\ D_{\ell q-1}(t + k\ \frac{p}{q}).$$

Note that for $0 < |k| \le \frac{q}{2}$ the inequality $|\widehat{\varphi}(k) - \frac{1}{2\pi i k}| \le \frac{C}{q}$ holds and therefore we may replace this infinite sum by

$$\sum_{\substack{-q/2<k\le q/2\\ k\ne 0}} \frac{1}{2\pi i k}\ D_{\ell q-1}(t + k\ \frac{p}{q})$$

(the $L^1$-norm of the error term may be estimated from above by

$$\Bigg(|\widehat{\varphi}(0)| + \sum_{0<|k|\le q/2} \Big|\widehat{\varphi}(k) - \frac{1}{2\pi i k}\Big| + \sum_{|k|\ge q/2} |\widehat{\varphi}(k)|\Bigg) \int_0^1 |D_{\ell q-1}(t)|\ dt = O(\log q)\ ).$$

Note also that $D_{\ell q-1}(t) = \frac{e^{2\pi i \ell q t}-1}{e^{2\pi i t}-1}$ and therefore, denoting $F(t) \overset{\mathrm{def}}{=} \frac{1}{e^{2\pi i t}-1}$, we can write the last expression in the form

$$(e^{2\pi i \ell q t} - 1) \sum_{\substack{-q/2<k\le q/2\\ k\ne 0}} \frac{1}{2\pi i k}\ F(t + k\ \frac{p}{q}).$$

Let $f(t)$ be the 1-periodic function defined on an interval

$$\Delta_j \stackrel{\text{def}}{=} \left(\frac{j}{q}-\frac{1}{2q}\ ,\ \frac{j}{q}+\frac{1}{2q}\right) \qquad \text{by} \qquad f(t)=\begin{cases} \frac{1}{2\pi i(t-j/q)} & \text{when } \ q \mid j \\ F(j/q) & \text{otherwise.} \end{cases}$$

It is easy to check that $\int_0^1 |F(t)-f(t)| \le 10$, say, and thereby we may replace $F(t)$ by $f(t)$, the $L^1$-norm of the corresponding difference being less than $20 \sum\limits_{0<|k|\le q/2} \frac{1}{2\pi |k|} = O(\log q)$. Thus we get

$$I_{\ell q-1}(\frac{p}{q}) = \int_0^1 \left|e^{2\pi i \ell q t}-1\right| \left| \sum_{\substack{-q/2<k\le q/2 \\ k\neq 0}} \frac{1}{2\pi i k}\ f(t+k\ \frac{p}{q})\right|\, dt + O(\log q)$$

$$= \sum_{j=0}^{q-1} \int_{\Delta_j} \ldots + O(\log q).$$

Consider each interval $\Delta_j$ separately. Note that the function $f(t+k\ \frac{p}{q})$ is constant on $\Delta_j$ unless $j+kp\equiv 0(\operatorname{mod} q)$ and that for each $j=0,\ldots,q-1$ there is exactly one solution of this comparison which lies between $-q/2$ and $q/2$. Therefore, substituting $\beta = q(t-j/q)$, we obtain

$$\int_{\Delta_j} \ldots = \int_{-1/2}^{1/2} \left| e^{2\pi i \ell \beta}-1\right| \left| -\frac{1}{4\pi^2 k_j}\ \frac{1}{\beta}+b_j\right|\, d\beta$$

where $b_j$ is some constant depending on $q$ and $j$ only (not on $\ell$) (this is true for $j\neq 0$, when we have even simpler representation without $-\frac{1}{4\pi^2 k_j}\ \frac{1}{\beta}$ under the absolute value sign).

As $|e^{2\pi i\ell\beta}-1|$ is an even function of $\beta$, we have

$$\int_{\Delta_j} \ldots \ge \int_{-1/2}^{1/2} |b_j|\ \left|e^{2\pi i \ell\beta}-1\right|\ d\beta = \frac{4}{\pi}\ |b_j|$$

and therefore

$$\sum_{j=0}^{q-1} |b_j| \le \frac{\pi}{4}\ I_{q-1}(\frac{p}{q}) + O(\log q) \le C\log^2 q$$

(this can be also checked by a direct computation of $b_j$).

As $|x+y|$ and $|x|+|y|$ may differ at most by $2\min(|x|,|y|)$, we get

$$\int_{\Delta_j} \ldots = \int_{-1/2}^{1/2} \frac{1}{4\pi^2 |k_j|}\ \frac{|e^{2\pi i\ell\beta}-1|}{|\beta|}\ d\beta + \int_{-1/2}^{1/2} |b_j|\ |e^{2\pi i \ell\beta}-1|\ d\beta$$

$$+ O\Big(\int_{-1/2}^{1/2} \min\Big(\frac{1}{|k_j|\ |\beta|}\ ,\ |b_j|\Big)\Big).$$

The first term is equal to $\frac{1}{|k_j|}\ \left(\frac{2}{\pi^3}\ \log\ell + O(1)\right)$, the second equals to $\frac{4}{\pi}\ |b_j|$ and at last

$$\int_{-1/2}^{1/2} \min\left(\frac{1}{|k_j|\ |\beta|}\ ,\ |b_j|\right) \le \int_{-1}^{1} \ldots = 2\ \frac{1}{|k_j|}\ \left(1 + \log^+\left(|k_j|\ |b_j|\right)\right).$$

Note that $k_j$ takes all values between $-q/2$ and $q/2$ except 0 when $j$ goes from 1 to $q-1$ and therefore

$$I_{\ell q-1}(\frac{p}{q}) = \frac{2}{\pi^3}\ \left(\sum_{0<|k|\le q/2} \frac{1}{|k|}\right)\ \log\ell\ + \ \frac{4}{\pi}\ \sum_{j=0}^{q-1}\ |b_j| + O(\log q)$$
$$+\ O\left(\sum_{j=1}^{q-1}\ \frac{1}{|k_j|}\ \ \log^+\left(|k_j|\ |b_j|\right)\right).$$

The first term may be also written in the form

$$\left(\frac{4}{\pi^3}\ \log q + O(1)\right)\ \log\ell = \frac{4}{\pi^3}\ \log q\ \ \log\ell + O(\log q)$$

(for $\log\ell \le (M-2)\log q$) and therefore to prove the "linearity" desired it is enough to show that the sum $\sum\limits_{j=1}^{q-1} \frac{1}{|k_j|}\ \log^+\left(|k_j|\ |b_j|\right)$ is essentially less than $\log^2 q$.

Using the inequality $\log^+ x \le \varepsilon x + 2\log 1/\varepsilon$ (to check it consider separately the cases $0 \le x \le 1/\varepsilon^2$ and $x \ge 1/\varepsilon^2$), we obtain that this sum is not greater than

$$\varepsilon \sum_{j=1}^{q-1}\ |b_j| + 2\log 1/\varepsilon\ \sum_{j=1}^{q-1}\ \frac{1}{|k_j|} \le C(\varepsilon\log^2 q + \log 1/\varepsilon\ \ \log q)$$

for every $\varepsilon > 0$. Putting $\varepsilon = \frac{1}{\log q}$, we get the estimate of order $\log q\ \ \log\log q$, completing the proof of (4).

## §6. **What to do with an arbitrary polygon.**

Consider any closed polygon $W \subset \mathbb{R}^2$ and choose some horizontal line $y = n_0 \in \mathbb{Z}$ which lies below it. For every non-vertical side $\Gamma = \{(x,y) \in \mathbb{R}^2 : a \le x \le b\ ,\ y = \alpha x + \nu\}$ of $W$ we are able to construct the corresponding trapezoid $T(\Gamma) \stackrel{\text{def}}{=} \{(x,y) \in \mathbb{R}^2 : a \le x \le b\ ,\ n_0 \le y \le \alpha x + \nu\}$. It is clear that

$$\chi_W = \sum_\Gamma \varepsilon(\Gamma)\ \chi_{T(\Gamma)} + \sum_{\Gamma:\varepsilon(\Gamma)=-1} \chi_\Gamma$$

($\chi_E$ denotes the characteristic function of the set $E$; $\varepsilon(\Gamma) = \pm 1$ depending on whether $W$ and $T(\Gamma)$ lie on the same side of $\Gamma$ or not; and, at last, the idle term $\sum\limits_{\Gamma:\varepsilon(\Gamma)=-1} \chi_\Gamma$ is due to our having assumed both $W$ and $T(\Gamma)$ closed).

Thus (in natural notations)

$$\mathcal{D}_{W,\lambda} = \sum_\Gamma \varepsilon(\Gamma)\ \mathcal{D}_{T(\Gamma),\lambda} + \sum_{\varepsilon(\Gamma)=-1} D_{\Gamma,\lambda} \quad , \quad \widetilde{\mathcal{D}}_{W,\lambda} = \sum_\Gamma \varepsilon(\Gamma)\ \widetilde{\mathcal{D}}_{T(\Gamma),\lambda}.$$

The kernels $D_{\Gamma,\lambda}$ are actually one-dimensional Dirichlet's kernels of order $\lambda$ or less and their contribution may be neglected. The repetition of the argumentation used in §1 gives

$$\mathcal{D}_{T(\Gamma),\lambda} = \widetilde{\mathcal{D}}_{T(\Gamma),\lambda} - e^{2\pi i \nu_\Gamma \lambda s} \sum_{a_\Gamma \lambda \le m \le b_\Gamma \lambda} \{\alpha_\Gamma\ m + \lambda\ \nu_\Gamma\}\ e^{2\pi i m (t + \alpha_\Gamma s)} + \text{“small terms”}.$$

Summarizing over all $\Gamma$, we get

$$\mathcal{D}_{W,\lambda} = \widetilde{\mathcal{D}}_{W,\lambda} - \sum_\Gamma \varepsilon(\Gamma)\ e^{2\pi i \nu_\Gamma \lambda s} \sum_{a_\Gamma \lambda \le m \le b_\Gamma \lambda} \{\alpha_\Gamma\ m + \lambda\ \nu_\Gamma\}\ e^{2\pi i m (t + \alpha_\Gamma s)} + \text{“small terms”}.$$

It is easy to check that all results of §2 remain valid for the (more general) sum $\sum\limits_{a\lambda \le v \le b\lambda} \{\alpha v + \theta\}\ e^{2\pi i v t}$. The only difference is that when we try to replace $\{\alpha v + \theta\}$ by $\{\frac{p}{q} v + \theta\} + \gamma v$, we get several (not more than $1 + |a| + |b|$) Dirichlet's kernels in the error term instead of one that; and the equality $\{\frac{p}{q} v + \theta\} = \varphi\{\frac{p}{q}\ v + \theta)$ doesn't hold precisely, but gives a Dirichlet kernel as an error as well.

Therefore we are able again to separate the contribution of $\widetilde{\mathcal{D}}_{W,\lambda}$ into $L_\lambda$ from that of the sum $\sum\limits_\Gamma \ldots$. The property (5) (together with the simple observation that for every 1-periodic function $g$ and every measurable set $E \subset (0,1)$ the inequality

$$\int_{-1/2}^{1/2} \Big( \int_{E+\alpha' s} |g(t + \alpha'' s|\ dt \Big)\ ds \le \frac{1 + |\alpha' - \alpha''|}{|\alpha' - \alpha''|}\ \mu(E) \int_0^1 |g(t)|\ dt$$

holds) allows also to separate the contributions of sides with different slopes.

Finally we get

$$L_\lambda = \iint\limits_{[-1/2,1/2]^2} |\widetilde{\mathcal{D}}_{W,\lambda}| \; dt \; ds$$
$$+ \sum_{\alpha} \iint\limits_{[-1/2,1/2]^2} \left| \sum_{\Gamma:\alpha_\Gamma=\alpha} \varepsilon(\Gamma) \; e^{2\pi i \nu_\Gamma \lambda s} \sum_{a_\Gamma \lambda \le m \le b_\Gamma \lambda} \{\alpha \; m + \lambda \; \nu_\Gamma\} \; e^{2\pi i m(t+\alpha s)} \right| \; dt \; ds$$
$$+ \; o(\log^2 \lambda).$$

Note also that the same is true if we replace the integration over $[-1/2,1/2]^2$ by that over $\Delta_1 \times \Delta_2$ where $\Delta_1$ and $\Delta_2$ are some intervals (with $\iint\limits_{\Delta_1\times\Delta_2} |\widetilde{\mathcal{D}}_{W,\lambda}| \; dt \; ds$ instead of $L_\lambda$ in the left hand side, of course).

Consider now one fixed group of parallel sides (of slope $\alpha$, say). Clearly, we may take $t+\alpha s$ as a new variable (which we shall denote by the same letter $t$). The only difficulty, when returning to old variables $t, s$, is that the "equidistribution property"

$$\iint\limits_{\Delta_1\times\Delta_2} \ldots = \mu(\Delta_1 \times \Delta_2) \iint\limits_{[-1/2,1/2]^2} \ldots + o(\log^2 \lambda)$$

in new variables will imply that in old variables for slant parallelograms instead of rectangles. But if this property is true for all parallelograms with some fixed slope of sides, it holds for every set measurable in Gordan sense as well! So, we are to investigate the sum

$$\sum_{\Gamma:\alpha_\Gamma=\alpha} \varepsilon(\Gamma) \; e^{2\pi i \nu_\Gamma \lambda s} \; f_{\Gamma,\lambda}(t)$$

where

$$f_{\Gamma,\lambda}(t) = \sum_{a_\Gamma \lambda \le m \le b_\Gamma \lambda} \{\alpha \; m + \lambda \; \nu_\Gamma\} \; e^{2\pi i m t}.$$

We need the following simple fact from the theory of almost-periodic functions

**Fact.**

Fix some finite set of real exponents $\nu_1, \ldots, \nu_\ell$. For every $\varepsilon > 0$ one can find $T_0 = T_0(\varepsilon; \nu_1, \ldots, \nu_\ell) > 0$ such that for any exponential polynomial $g(s) = \sum_{j=1}^{\ell} c_j \; e^{2\pi i \nu_j s}$ and for any interval $\Delta \subset \mathbb{R}$ of length greater than $T_0$ the inequality

$$\left| \frac{1}{\mu(\Delta)} \int_\Delta |g(s)| \; ds - \mathcal{M}(|g|) \right| \le \varepsilon \sum_{j=1}^{\ell} |c_j|$$

holds, where

$$\mathcal{M}(|g|) \stackrel{\text{def}}{=} \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |g(s)|\; ds$$

is the "invariant mean" of the function $|g|$ over $\mathbb{R}$.

**Proof.** (well-known, of course)

Without loss of generality $\sum\limits_{j=1}^{\ell} |c_j| = 1$. The existence of the invariant mean and the analogue of the inequality desired is evident for $|g(s)|^2$ , $|g(s)|^4, \ldots$ instead of $|g(s)|$ just because they are exponential polynomials in fixed sets of exponents with bounded coefficients. It remains only to note that $|x|$ can be uniformly approximated on $[-1,1]$ by finite linear combinations of $|x|^2$ , $|x|^4, \ldots$.

As $g$ is determined by its coefficients, we may also denote $\mathcal{M}(|g|)$ by $\mathcal{M}(c_1, \ldots, c_\ell)$. It follows immediately from the fact above that for every interval $\Delta_2 \subset \mathbb{R}$

$$\int_{\Delta_2} \Big|\sum_{\Gamma} e^{2\pi i\nu_\Gamma \lambda s}\; f_{\Gamma,\lambda}(t)\Big|\; ds = \mu(\Delta_2)\; \mathcal{M}\big(\{f_{\Gamma,\lambda}(t)\}\big) + o\Big(\sum_{\Gamma} |f_{\Gamma,\lambda}(t)|\Big)$$

when $\lambda \to \infty$ ($\mathcal{M}$ corresponds to the set $\{\nu_\Gamma\}_{\Gamma:\alpha_\Gamma=\alpha}$ of exponents; $\{\ldots\}$ is here the "set sign", not fractional part, of course). Then

$$\iint\limits_{\Delta_1\times\Delta_2} \Big|\sum_{\Gamma} e^{2\pi i\nu_\Gamma \lambda s}\; f_{\Gamma,\lambda}(t)\Big|\; dt\; ds = \mu(\Delta_2)\int_{\Delta_1} \mathcal{M}\big(\{f_{\Gamma,\lambda}(t)\}\big)\; dt + o(\log^2\lambda),$$

for (as it was mentioned above)

$$\int_0^1 |f_{\Gamma,\lambda}(t)|\; dt = O(\log^2\lambda).$$

Using again the auxiliary function $\psi(s)$ (1-periodic continuation of $s(e^{2\pi is}-1)$ from $[0,1]$), we get that

$$\int_0^1 \big|f_{\Gamma,\lambda}(t+\alpha) - e^{-2\pi i\nu_\Gamma\lambda}\; f_{\Gamma,\lambda}(t)\big|\; dt \le C\log\lambda$$

and therefore

$$\int_{\Delta_1+\alpha} \mathcal{M}\big(\{f_{\Gamma,\lambda}(t)\}\big)\; dt = \int_{\Delta_1} \mathcal{M}\big(\{e^{-2\pi i\nu_\Gamma\lambda}\; f_{\Gamma,\lambda}(t)\}\big)\; dt + O(\log\lambda).$$

To establish the equidistribution property desired, it remains only to note that for every $\xi \in \mathbb{R}$ and $\{c_\Gamma\} \subset \mathbb{C}$

$$\mathcal{M}(\{e^{-2\pi i\nu_\Gamma\xi}\; C_\Gamma\}) = \mathcal{M}(\{C_\Gamma\})$$

just because the "invariant mean" is invariant under the shift of variable by $\xi$.

To prove the good analogue of "Proposition", it is more convenient to keep both variables $s$ and $t$, not replacing integral over $s$ by the corresponding invariant mean.

We shall again use the approximation

$$\sum_{k\in\mathbb{Z}} \widehat{\varphi}(k)\ e^{2\pi i k\nu_\Gamma\lambda} \sum_{a_\Gamma\lambda\le m\le b_\Gamma\lambda} e^{2\pi i m(t+k\alpha)}$$

for $f_{\Gamma,\lambda}(t)$ and, thereby, the approximation

$$\sum_{k\in\mathbb{Z}} \widehat{\varphi}(k) \sum_{\Gamma} \varepsilon(\Gamma)\ e^{2\pi i\nu_\Gamma\lambda(s+k)} \sum_{a_\Gamma\lambda\le m\le b_\Gamma\lambda} e^{2\pi i m(t+k\alpha)}$$

for the sum to be investigated. Choose some integer $\ell \in \big(\frac{\lambda}{\log\lambda}\ ,\ \frac{2\lambda}{\log\lambda}\big)$, say, and split the interval $(-1/2,1/2)$ into $2\ell+1$ subintervals

$$\Delta_j \overset{\text{def}}{=} \big\{t : \big|t-\frac{j}{2\ell+1}\big| \le \frac{1}{4\ell+2}\big\} \quad , \quad j=-\ell,\ldots,\ell.$$

Define the function

$$F:\big(-\frac{1}{2}\ ,\ \frac{1}{2}\big)\to\mathbb{C} \quad \text{by} \quad F(t)=\begin{cases} 0 & ,\quad t\in\Delta_0 \\ \frac{1}{2\pi i}\ \frac{2\ell+1}{j} & ,\quad t\in\Delta_j \quad ,\quad j\neq 0 \end{cases}$$

(the concrete expression for $F(t)$ doesn't matter; it is important only that it approximates the denominator $\frac{1}{e^{2\pi i t}-1}$ of the Dirichlet kernel "in $L^1$-sense" and is constant on intervals essentially longer than $1/\lambda$). Denote by $D_\lambda(t,s)$ the 1-periodic continuation (with respect to $t$, not to $s$) of the function

$$F(t)\sum_{\Gamma}\varepsilon(\Gamma)\ e^{2\pi i\nu_\Gamma\lambda s}\ \big(e^{2\pi i b_\Gamma\lambda t}-e^{2\pi i a_\Gamma\lambda t}\big)$$

(defined initially on $(-1/2,1/2)\times\mathbb{R}$).

It is easy to check that the above sum may be replaced by $\sum_{k\in\mathbb{Z}} \widehat{\varphi}(k)\ D_\lambda(t+\alpha k, s+k)$ with an error about $\log\lambda\ \log\log\lambda$ (in $L^1$-norm). Note that for every partition interval $\Delta_j$ and for every $k\in\mathbb{Z}$ we have

$$\iint_{\Delta_j\times[-1/2,1/2]} |D_\lambda(t,s+k)|\ dt\ ds$$

$$=\big|F\big(\frac{j}{2\ell+1}\big)\big| \iint_{\Delta_j\times([-1/2,1/2]+k)} \Big|\sum_{\Gamma}\varepsilon(\Gamma)\ e^{2\pi i\nu_\Gamma\lambda s}\ \big(e^{2\pi i b_\Gamma\lambda t}-e^{2\pi i a_\Gamma\lambda t}\big)\Big|\ dt\ ds$$

$$=\mu(\Delta_j)\ \big|F\big(\frac{j}{2\ell+1}\big)\big|\ (\mathcal{M}_\alpha+o(1))$$

$$=(\mathcal{M}_\alpha+o(1))\int_{\Delta_j}|F(t)|\ dt \qquad (\lambda\to+\infty)$$

where $\mathcal{M}_\alpha$ is the invariant mean of the exponential polynomial (in two variables)

$$\mathcal{G}_\alpha(t,s) \stackrel{\text{def}}{=} \sum_{\Gamma:\alpha_\Gamma=\alpha} \varepsilon(\Gamma)\ e^{2\pi i\nu_\Gamma s}\ \left(e^{2\pi i b_\Gamma t} - e^{2\pi i a_\Gamma t}\right),$$

i.e.

$$\mathcal{M}_\alpha = \lim_{T_1,T_2\to\infty} \frac{1}{4T_1T_2} \iint\limits_{[-T_1,T_1]\times[-T_2,T_2]} |\mathcal{G}_\alpha(t,s)|\ dt\ ds$$

(the existence of $\mathcal{M}_\alpha$ and the "uniform convergence" property used can be proved in two dimensions just by a repetition of one-dimensional arguments above).

It follows from here immediately that for each $\xi \geq \frac{1}{\sqrt{\lambda}}$, say,

$$\iint\limits_{[-\xi,\xi]\times[-1/2,1/2]} |D_\lambda(t,s+k)|\ dt\ ds = (\mathcal{M}_\alpha + o(1))\ \frac{1}{\pi}\ \log\lambda\xi.$$

Together with the obvious estimate $D_\lambda(t,s+k) \leq \frac{C}{t}$ $(-1/2 \leq t \leq 1/2)$ this allows to carry out all considerations of §§3-5 and obtain an analogue of the "Proposition" in which all absolute constants are replaced by "absolute multiples of $\mathcal{M}_\alpha$" (i.e. we write $c\ \mathcal{M}_\alpha$ instead of $c$).

From other side (see [] for details) it is possible to separate the contributions of sides with different slopes into the asymptotics of $\iint\limits_{[-1/2,1/2]^2} |\widetilde{\mathcal{D}}_\lambda(t,s)|\ dt\ ds$, that of sides with slope $\alpha$ being precisely $\frac{1}{2\pi^2}\ \mathcal{M}_\alpha\ \log^2\lambda$.

Thus the interference of the contributions of different parallel sides into the "boundary term" is in essential the same as the interference of their contributions into the "integral term" $\iint\limits_{[-1/2,1/2]} |\widetilde{\mathcal{D}}_\lambda|\ dt\ ds$ and (after bringing all obtained results together and simple reformulations) we get the following

**Proposition 2.**

Let $W$ be an arbitrary polygon with slopes of sides $\alpha_1,\ldots,\alpha_d$ (for vertical sides we assume $\alpha = \infty \in \mathbb{Q}$)

**I**)

$$\iint\limits_{[-1/2,1/2]^2} |\widetilde{\mathcal{D}}_\lambda|\ dt\ ds = \frac{1}{2\pi^2}\left(\sum_{j=1}^{d}\mathcal{M}_{\alpha_j} + o(1)\right)\ \log^2\lambda \qquad (\lambda\to\infty)$$

**II**)

$$L_\lambda = \iint\limits_{[-1/2,1/2]^2} |\widetilde{\mathcal{D}}_\lambda| \; dt \; ds + \sum_{j=1}^{d} \mathcal{M}_{\alpha_j} \; I'_\lambda(\alpha_j) + o(\log^2 \lambda)$$

where

$$I'_\lambda(\alpha_j) \overset{\text{def}}{=} \frac{1}{\mathcal{M}_{\alpha_j}} \iint\limits_{[-1/2,1/2]^2} \left| \sum_{\Gamma:\alpha_\Gamma=\alpha_j} \varepsilon(\Gamma) \; e^{2\pi i \nu_\Gamma \lambda s} \sum_{a_\Gamma \lambda \le m \le b_\Gamma \lambda} \{\alpha \; m + \lambda \; \nu_\Gamma\} \; e^{2\pi i m t} \right| \; dt \; ds$$

satisfy (1)-(4) and two-dimentional analog of (5).

**III**)

$$\overline{\lim_{\lambda\to\infty}} \; \frac{1}{\log^2 \lambda} \left( L_\lambda - \iint\limits_{[-1/2,1/2]^2} |\widetilde{\mathcal{D}}_\lambda| \; dt \; ds \right) > 0$$

unless all $\alpha_j \in \mathbb{Q}$ and

$$\underline{\lim}_{\lambda\to\infty} \; \frac{1}{\log^2 \lambda} \left( L_\lambda - \iint\limits_{[-1/2,1/2]^2} |\widetilde{\mathcal{D}}_\lambda| \; dt \; ds \right) = 0$$

if and only if $\alpha_1, \dots, \alpha_d$ can be very good approximated by rationals simultaneously, namely, if for every $M > 0$ there exist $q \in \mathbb{N}$; $p_1, \dots, p_d \in \mathbb{Z}$ such that $|\alpha_j - \frac{p_j}{q}| \le \frac{1}{q^M}$ for all $j = 1, \dots, d$ (and, of course, $q \ge 2$).

In particular we obtain that for large $\lambda \ge \lambda_0(W)$

$$c \iint\limits_{[-1/2,1/2]^2} |\widetilde{\mathcal{D}}_\lambda| \; dt \; ds \le \iint\limits_{[-1/2,1/2]^2} |\mathcal{D}_\lambda| \; dt \; ds \le C \iint\limits_{[-1/2,1/2]} |\widetilde{\mathcal{D}}_\lambda| \; dt \; ds$$

with some absolute constants $c$ and $C$.

An interesting unsolved problem is whether the same is true for arbitrary not very bad (convex, say) domains $W$, not only for polygons.