Counterexample to the infinite dimensional Carleson embedding theorem

Fedor Nazarov, Serguei Treil, and Alexander Volberg *

Abstract. We are going to show that the classical Carleson embedding theorem fails for Hilbert space valued functions and operator measures.

1 Formulation of results.

Let $\mathbb{T}$ be the unit circle, and let $\mathcal{D}$ be the collection of its dyadic arcs. H stands for a Hilbert space, and let $(\ ,\ )$ denote its inner product. Let m denote the normalized ($m(\mathbb{T}) = 1$) Lebesgue measure on $\mathbb{T}$. Let μ be a positive operator-valued Borel measure in the unit disc $\mathbb{D}$. In particular $\mu(E)$ is a nonnegative operator on H for an arbitrary Borel measurable subset of $\mathbb{D}$.

For an arc I consider the Carleson box $Q_I = \{z \in \mathbb{D} : z/|z| \in I, 1 - |I| \le |z| < 1\}$ built on I in $\mathbb{D}$, and let T_I denote the half of Q_I, for which $|z| < 1 - |I|/2$. Let $|I|$ denote the length of I. Consider the embedding operator: $f \overset{A}{\mapsto} \sum_{I\in\mathcal{D}} \left(|I|^{-1} \int_I f\, dm\right) \chi_{T_I}(z)$, $f \in L^2(dm)$. We are interested in the question, when is the operator A from $L^2(H; dm)$ to $L^2(H;\mu)$ bounded, i.e., when do we have

$$\|Af\|_\mu \overset{\text{def}}{=} \left(\int_{\mathbb{D}} \big(d\mu(z)(Af)(z), (Af)(z)\big)_H\right)^{1/2} \le C\|f\|_{L^2} = \left(\int_{\mathbb{T}} \|f\|^2 dm\right)^{1/2} \qquad \forall f \in L^2(H;\mu)\,?$$

In general, integration over an operator valued measure μ is a delicate topic. At issue is the meaning to the integral $\int_{\mathbb{D}} \big(d\mu(z)(Af)(z), (Af)(z)\big)_H$. However, in our case the function Af is constant on every T_I, so the integral is just the sum

$$\sum_{I\in\mathcal{D}} \big(\mu(T_I)\langle f\rangle_I, \langle f\rangle_I\big),$$

where $\langle f\rangle_I := |I|^{-1} \int_I f dm$.

The classical Carleson embedding theorem (see [1]) implies (actually it is equivalent to) the following result in the dyadic setting.

Theorem A. *In the scalar case* ($\dim H = 1$) *the operator A is bounded from $L^2(dm)$ to $L^2(\mu)$ if and only if*

$$\mu(Q_J) \le C|J| \qquad \text{for every } J \in \mathcal{D}.$$

Moreover, there exists an absolute constant K such that $\|A\| \le K\sqrt{C}$.

*Partially supported by the grant from the National Science Foundation DMS9622936.

Corollary B. *In the scalar case, the embedding operator is bounded from $L^2(\mathbb{T})$ to $L^2(\mu)$ if and only if it is uniformly bounded on all test functions $f = \chi_J$, $J \in \mathcal{D}$.*

In fact, $|J| = \|\chi_J\|^2_{L^2(dm)}$; $\mu(Q_J) \le \|T\chi_J\|^2_{L^2(\mu)}$.

The embedding theorems for the operator A can serve as simple models of embedding theorems for the operator $\mathcal{H}$ of harmonic extension. The matrix and operator-valued embedding theorems for $\mathcal{H}$ play an important role in Weighted Norm Inequalities with operator weights, in the Operator Corona Problem, and in Control Theory.

In this note we make a nearly sharp dimensional estimates for the norm of embedding operator. The estimate from below leads to a counterexample to the infinite dimensional Carleson embedding theorem. Estimates from above are stated mostly for the sake of completeness. To formulate our results we need the notion of Carleson intensity of an operator-valued measure.

Carleson intensity of measure. Given an operator valued measure μ on $\mathbb{D}$ we call the following number $\|\mu\|_C$ its Carleson intensity:

$$\|\mu\|_C^2 \stackrel{\text{def}}{=} \sup \frac{1}{|I|}\|\mu(Q_I)\| = \sup_{I\in\mathcal{D}} \sup_{e\in H:\|e\|=1} \frac{1}{|I|}(\mu(Q_I)e, e)\,.$$

Theorem A claims that there exists a universal constant K such that (in the scalar case) the norm of the operator $A : L^2(dm) \to L^2(\mu)$ satisfies $\|A\| \le K\|\mu\|_C$.

If $\dim H = n$, one can trivially get the estimate $\|A\| \le \sqrt{n}K\|\mu\|_C$.

The following two theorems are the main results of this note.

Theorem 1.1 *Let $\dim H = n$. For any operator valued measure μ the following inequality holds*

$$\|A\|/\|\mu\|_C \le C(\log n)\,.$$

Here $C < \infty$ is an absolute constant.

Theorem 1.2 *Let $\dim H = n$. There exists an operator valued measure μ such that*

$$\|A\|/\|\mu\|_C \ge c(\log n)^{1/2}\,.$$

Here $c > 0$ is an absolute constant.

Corollary 1.3 *There exists a positive operator valued measure in infinite dimensional Hilbert space with finite Carleson intensity that does not allow bounded embedding.*

Theorem 1.1 was proven independently by Nets Hawk Katz in [2]. His argument is based on an ingenious stopping time procedure.

2 Proof of Theorem 1.2.

The sought after operator valued measure μ in $\mathbb{D}$ will have the form

$$d\mu = \sum_{I:I\in\mathcal{D},|I|\ge 2^{-n}} (\cdot, \varphi_I)\varphi_I |I| \delta_{c_I}\,,$$

where c_I is the center of T_I, δ_{c_I} is the unit point mass at c_I, and the φ_I are certain vectors in H to be chosen later. To construct φ_I, we first choose a sequence $\{a_j\}_{j=0}^{\infty}$ of positive numbers (again to be fixed later) such that

$$\sum (j+1)a_j^2 \leq 1\,. \tag{2.1}$$

We define the rank of a dyadic arc J of length 2^{-j} by $\mathrm{rk}(J) = j$. We then define the relative rank of the dyadic arcs J, I of lengths $2^{-j}, 2^{-i}$ by $\mathrm{rk}(J:I) = |j-i|$. Also r_k denotes the k^{th} Rademacher function, $r_k(e^{2\pi i t}) = (-1)^{\lfloor 2^k t\rfloor}$, $t \in [0,1)$; here $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x. We write $r_k(z) = \sum_{I:\,\mathrm{rk}(I)=k} \varepsilon_I \chi_I(z)$, with $\varepsilon_I = \pm 1$. Let $\{e_m\}_{m=0}^{n}$ be a fixed orthonormal basis in $H, \dim H = n+1$. Here is the choice of φ_I (x denotes an arbitrary point of I):

$$\varphi_I = \sum_{J:I\subset J} a_{\mathrm{rk}(J:I)}\varepsilon_J e_{\mathrm{rk}(J)} = \sum_{j=0}^{i} a_{i-j} r_j(x) e_j\,. \tag{2.2}$$

2.1 Estimate for the Carleson intensity from above.

Given a vector $e \in H, e = \sum_{j=0}^{n} b_j e_j$, and a dyadic arc K, we have to estimate $\sum_{I:I\subset K} |(e,\varphi_I)|^2|I|$. Fix the arc K and denote by P the orthogonal projection in H onto the linear span of $e_m, m \leq k \stackrel{\mathrm{def}}{=} \mathrm{rk}(K)$. One can observe that $P\varphi_I = \sum_{J:K\subset J} a_{\mathrm{rk}(J:I)}\varepsilon_J e_{\mathrm{rk}(J)} = \sum_{j=0}^{k} a_{i-j} r_j(x) e_j$ depends only on $i \stackrel{\mathrm{def}}{=} \mathrm{rk}(I)$, when $I \subset K$, i.e. $P\varphi_I = f_i$, for $I \subset K$. Using this we can write

$$\phi_i(x) \stackrel{\mathrm{def}}{=} \sum_{I\subset K,\,\mathrm{rk}(I)=i} (e,\varphi_I)\chi_I(x) = \sum_{\substack{I,J:\,\mathrm{rk}(I)=i\\ I\subset J\subsetneq K}} a_{\mathrm{rk}(J:I)} b_{\mathrm{rk}(J)} r_{\mathrm{rk}(J)}(x)\chi_I(x) + (e,f_i)\chi_K(x) =$$

$$= \left(\sum_{j=k+1}^{i} a_{i-j} b_j r_j(x) + (e,f_i)\right)\chi_K(x)$$

The orthogonality of the Rademacher functions on K and the identity $\int_{\mathbb{T}} \phi_i(x)^2 dm = \sum_{I:\mathrm{rk}(I)=i, I\subset K} (e,\varphi_I)^2|I|$ show that after summing by i

$$\sum_{I:\,I\subset K} (e,\varphi_I)^2|I| = |K|\sum_{i=k+1}^{n}\sum_{j=k+1}^{i} a_{i-j}^2 b_j^2 + |K|\sum_{i\geq k}(e,f_i)^2 = \Sigma_1 + \Sigma_2\,.$$

The sums can be estimated using (2.1) as follows

$$\Sigma_1 = |K|\sum_{j=k+1}^{n} b_j^2 \sum_{i=j}^{n} a_{i-j}^2 \leq |K| \sum_{j=k+1}^{n} b_j^2 (\sum_{m=0}^{n} a_m^2) = (\sum_{m=0}^{n} a_m^2)(\sum_{j\geq k+1} b_j^2)|K| \leq \|e\|^2|K|\,,$$

$$\Sigma_2 \leq \|e\|^2|K|\sum_{j\geq k}\|f_j\|^2 = \|e\|^2|K|\sum_{j=k}^{n}\sum_{s=0}^{k} a_{j-s}^2 =$$

$$= \|e\|^2|K|\sum_{\ell=0}^{n-k}\sum_{i=\ell}^{k+\ell} a_i^2 \leq \|e\|^2|K|\sum_{\ell=0}^{n-k}\sum_{i=\ell}^{n} a_i^2 \leq \|e\|^2|K|\sum_{i=0}^{n}(i+1)a_i^2 \leq \|e\|^2|K|\,.$$

Thus, (2.1) implies that the Carleson intensity of measure μ, which is built with the help of φ_I from (2.2), is bounded by $\sqrt{2}$.

2.2 Estimate for the embedding constant from below.

We now build an H-valued function $E(t)$, $t \in \mathbb{T}$, such that $\int_{\mathbb{T}} \|E(t)\|^2_H = n+1$ and $\int_{\mathbb{D}} (d\mu\, AE, AE)_H = \sum_{I\in\mathcal{D}} |(\langle E\rangle_I, \varphi_I)_H|^2|I| \geq a_0^2 + (a_0+a_1)^2 + \cdots + (a_0+a_1+\cdots+a_n)^2$. Here $\langle E\rangle_I \in H$ naturally denotes the average $|I|^{-1}\int_I E(t)dm(t)$ of E over I. As $\int_{\mathbb{T}} \|E(t)\|^2_H dm(t) = \|E\|^2_{L^2(H;dm)}$ and $\int_{\mathbb{D}} (d\mu\, AE, AE) = \|AE\|^2_{L^2(H;\mu)}$, we have

$$\|A\| \geq \frac{1}{\sqrt{n+1}} \Big[\sum_{\ell=0}^{n} (\sum_{i=0}^{\ell} a_i)^2\Big]^{1/2}. \tag{2.3}$$

This estimate finishes the proof. Indeed, it is sufficient to choose $a_j = \frac{\beta}{j+1}$, with $\beta = c(\log n)^{-1/2}$ to have (2.1) satisfied, and to have the right part of (2.3) at least $c(\log n)^{1/2}$.

To choose E, we naturally follow the sign pattern for φ_I. Our E will be a step function of constant norm, $\|E(t)\|^2 \equiv n+1$. On each dyadic arc I, $|I| = 2^{-n}$, we define

$$E(t) \stackrel{\text{def}}{=} \sum_{k=0}^{n} \frac{(\varphi_I, e_k)}{|(\varphi_I, e_k)|} e_k = \sum_{k=0}^{n} r_k(t) e_k\,. \tag{2.4}$$

To estimate $\int_{\mathbb{D}} (d\mu\, AE, AE)_H = \sum_{J\in\mathcal{D}} |(\langle E\rangle_J, \varphi_J)|^2|J|$ from below, let us consider the following S-function:

$$S^2(E)(t) \stackrel{def}{=} \sum_{J: t\in J} |(\langle E\rangle_J, \varphi_J)|^2\,. \tag{2.5}$$

Actually it is identically constant. In fact, the pattern of signs for $\langle E\rangle_J$ is exactly the same as for φ_J. Namely, for any J such that $|J| = 2^{-\ell}$

$$(\langle E\rangle_J, \varphi_J) = a_0 + a_1 + \cdots + a_\ell\,. \tag{2.6}$$

Therefore, plugging (2.6) into (2.5), we get $S^2(E)(t) \equiv (a_0+a_1+\cdots+a_n)^2 + (a_0+a_1+\cdots+a_{n-1})^2 + \cdots + a_0^2$. On the other hand, we have the following standard formula: $\sum_{J\in\mathcal{D}} |(\langle E\rangle_J, \varphi_J)|^2|J| = \int_{\mathbb{T}} S^2(E)(t)dm(t)$, which is just the change of order of summation and integration. Thus

$$\int_{\mathbb{D}} (d\mu AE, AE)_H = \sum_{J\in\mathcal{D}} |(\langle E\rangle_J, \varphi_J)|^2|J| \geq \sum_{\ell=0}^{n} \left(\sum_{k=0}^{\ell} a_k\right)^2. \tag{2.7}$$

Thus Theorem 1.2 and also Corollary 1.3 are fully proven.

3 Sketch of the proof of Theorem 1.1.

A detailed proof of Theorem 1.1 can be found in [5] together with the explanation of how to guess the Bellman function for the problem.

Without loss of generality $\mu = \sum_{I\in\mathcal{D}} \mu_I |I| \delta_{c_I}$, where μ_I are $n\times n$ positive definite matrices and δ_{c_I} is the unit point mass at the center of T_I. Let us consider the operators $M_J \stackrel{\text{def}}{=} |J|^{-1}\sum_{I\subset J} \mu_I\, |I|$.

Let **Id** stand for the identity matrix. Assume that the Carleson intensity $\|\mu\|_C \le 1$, which means that in the sense of positive operators, $M_I \le \mathbf{Id}$. We need to prove that for any $f \in L^2(H; dm)$ with $\|f\| = 1$, the following inequality is satisfied

$$\sum_{I\in\mathcal{D}} (\mu_I \langle f\rangle_I, \langle f\rangle_I)|I| \le C(\log n)^2 . \tag{3.1}$$

Notice that the scalar measure $\nu \stackrel{\text{def}}{=} \operatorname{trace}\mu$ has the simple intensity estimate $\|\nu\|_C^2 \le n\|\mu\|_C^2 \le n$. Since $(\mu_I \langle f\rangle_I, \langle f\rangle_I) \le \operatorname{trace}\mu_I \,\langle\|f\|\rangle_I^2$, one can get the estimate in (3.1) with constant Cn instead of $C(\log n)^2$.

It is clear that the same estimate by Cn follows for the sum

$$h(z) \stackrel{\text{def}}{=} \sum_{I\in\mathcal{D}} ((\mathbf{Id} + zM_I)^{-1}\mu_I(\mathbf{Id} + zM_I)^{-1}\langle f\rangle_I, \langle f\rangle_I)|I|$$

for arbitrary $z \in \mathbb{C}, |z| \le 1/2$, because $\|(\mathbf{Id} + zM_I)^{-1}\| \le 2$ for $|z| \le 1/2$.

On the other hand we now obtain a completely different estimate for $h(x)$ for real $x \in (0, 1/2)$.

To this end, we need to consider the following Bellman functions ([3], [4])

$B_x(I) \stackrel{\text{def}}{=} B_x(\langle\|f\|^2\rangle_I, f_I, M_I) = \langle\|f\|^2\rangle_I - ((\mathbf{Id} + xM_I)^{-1}\langle f\rangle_I, \langle f\rangle_I)$.

It is obvious that $0 \le B_x(I) \le \langle\|f\|^2\rangle_I$ if $x \in (0, 1/2)$. On the other hand, $B_x(I)$ satisfies the following crucial concavity inequality (I_+, I_- are right and left halves of I respectively):

$$B_x(I) \ge x((\mathbf{Id} + xM_I)^{-1}\mu_I(\mathbf{Id} + xM_I)^{-1}\langle f\rangle_I, \langle f\rangle_I) + \frac{B_x(I_+) + B_x(I_-)}{2}, \tag{3.2}$$

which the reader can check by calculation. Let us continue (3.2) to the right by applying (3.2) to the halves I_+ and I_-, and then to $I_{++}, I_{+-}, I_{-+}, I_{--}$ etc. If the starting arc I is the whole $\mathbb{T}$, we will obtain

$$xh(x) \le \|f\|^2 \le 1 . \tag{3.3}$$

Now we gather the information. The function h, which is holomorphic on $\{z : |z| \le 1/2\}$, is bounded by Cn globally in $\{z : |z| \le 1/2\}$, and also satisfies (3.3) for $x \in (0, 1/2)$. Consider h as an analytic function in $\Omega := \frac{1}{2}\mathbb{D} \setminus [(\log n)^{-2}, 1/2]$. Note that $|h| \le (\log n)^2$ on $[(\log n)^{-2}, 1/2]$, and that we have the trivial estimate $|h| \le Cn$ on $\frac{1}{2}\mathbb{T}$. Note also that the harmonic measure of $\frac{1}{2}\mathbb{T}$ with respect to the region Ω and the point 0 is about $C/\log n$. Using the standard "two constant lemma" we get $h(0) \le C(\log n)^2$. But $h(0)$ is the left part of (3.1). So (3.1) is proven, and we are done with Theorem 1.1.

4 Open questions.

1. The growth of the norm of the embedding operator is between $c(\log n)^{1/2}$ and $C \log n$ for measure with intensity bounded by 1. What is the sharp dimensional estimate?

2. What are the results for the harmonic embedding instead of the dyadic embedding operator considered in this paper? We think that it is possible to prove similar estimates for the harmonic embedding operator.

3. In the case of the harmonic embedding of vector valued functions, one suspects that a measure can be an embedding measure for analytic vector valued functions but not for antianalytic vector valued functions. What is an example of such a measure?

4. How can one characterize measures, which provide an embedding for analytic vector valued functions only?

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