Complete version of the Turan lemma for trigonometric polynomials on the unit circumference.

§0. Introduction and preliminaries.

Let $p(z) = \sum_{k=1}^{n} c_k z^{m_k}$ ($c_k \in \mathbb{C}$, $m_1 < \cdots < m_n \in \mathbb{Z}$) be a trigonometric polynomial in n characters considered as a function on the unit circumference $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Well-known Turan's lemma claims that for every arc $E \subset \mathbb{T}$

$$||p||_{L^\infty(\mathbb{T})} \leqslant \left\{\frac{4e}{\mu(E)}\right\}^{n-1} ||p||_{L^\infty(E)}.$$

(as usual we denote by μ the Haar measure on $\mathbb{T}$ and for each subset $H \subset \mathbb{T}$ we define $||f||_{L^q(H)} \stackrel{\text{def}}{=} \left(\int_H |f|^q d\mu\right)^{\frac{1}{q}}$ for $0 < q < \infty$; $||f||_{L^\infty(H)} \stackrel{\text{def}}{=} \sup_{z\in H} |f(z)|$ and $||f||_{L^0(H)} \stackrel{\text{def}}{=} \exp\{\int_H log\,|f|\,d\mu\}$).

In my previous paper it was shown that the same estimate remains valid for an arbitrary measurable set $E \subset \mathbb{T}$ if we replace $4e$ by 14 in the numerator. For the decreasing rearrangement Φ of $|p|$ (defined as the (unique) continuous decreasing function $\Phi : [0,1] \to [0,+\infty)$ satisfying $\mu\{z \in \mathbb{T} : |p(z)| > y\} = \mu\{t \in [0,1] : \Phi(t) > y\}$ for every $y > 0$) it means that $\Phi(t) \geqslant \left(\frac{1-t}{14}\right)^{n-1}\Phi(0)$ and therefore for each set $E \subset \mathbb{T}$

$$||p||_{L^0(E)} \geqslant \exp\left\{\frac{1}{\mu(E)}\int_{1-\mu(E)}^{1} \log\,\Phi(t)\,dt\right\}$$
$$\geqslant \left(\frac{\mu(E)}{14e}\right)^{n-1}\Phi(0) = \left(\frac{\mu(E)}{14e}\right)^{n-1}||p||_{L^\infty(T)}.$$

Thus, for every $q \in [0,+\infty]$

$$||p||_{L^q(\mathbb{T})} \leqslant \left\{\frac{14e}{\mu(E)}\right\}^{n-1} ||p||_{L^q(E)}.$$

As the example of the polynomial $p(z) = (z-1)^{n-1}$ and a small arc $E \subset \mathbb{T}$ centered at 1 shows, this estimate cannot be essentially improved for sets E of small measure even if we place $L^0(\mathbb{T})$-norm to the left hand side and $L^\infty(E)$-norm to the right hand that. But for sets E close to the whole circumference $\mathbb{T}$ one could expect some inequality of the kind $||p||_{L^q(\mathbb{T})} \leqslant B(\mu(E))^{n-1}||p||_{L^q(E)}$. where $B(\mu) \to 1$ when $\mu \to 1-$, while the factor given by Turan lemma or by its generalization remains very far from 1 even for the whole circumference and this drawback cannot be removed by a minor modification of any proof known so far even for arcs E. The main aim of this paper is to confirm this common belief and to show that for every $0 \leqslant q \leqslant 2$ and each set $E \subset \mathbb{T}$ of measure $\mu(E) \geqslant \frac{1}{3}$ the estimate

$$||p||_{L^q(\mathbb{T})} \leqslant e^{A(n-1)\mu(\overline{E})}||p||_{L^q(E)} \qquad (*)$$

holds with some absolute constant $A > 0$ not depending on q (by $\overline{E}$ we denote the complement $\mathbb{T} \setminus E$ of the set E). It is not very difficult (but rather boring, so we aren't going to perform it here) to check that there is an absolute constant $a > 0$

such that for each $q \in [0,2]$, $n \in \mathbb{N}$ and for every arc $E \subset \mathbb{T}$ of measure $\mu(E) \geqslant \frac{1}{3}$ one can find a polynomial $p(z)$ in not more than n characters for which

$$||p||_{L^q(\mathbb{T})} \geqslant e^{a(n-1)\mu(\overline{E})}||p||_{L^q(E)}.$$

(the polynomial in the example can always be taken of the form $p(z) = \left(\frac{z^m-1}{z-1}\right)^r$ with some m, r satisfying $mr \leqslant n$). So, the estimate above is the best possible one up to the numerical value of the absolute constant involved. If $q > 2$, one can merely use the fact that $||p||_{L^\infty(\mathbb{T})} \leqslant \sqrt{n}||p||_{L^2(\mathbb{T})}$ to get the same estimate for "not extremaly large" sets E, namely for sets E of measure $\mu(E) \leqslant 1 - \frac{\log n}{n}$. If the measure of $\overline{E}$ is essentially less that $\frac{\log n}{n}$, the inequality (*) is no longer true for $q > 2$ and we'll return to the question what one should replace it with in the end of the paper.

§1. The proof of (*) for q=2.

Lemma 1.

For every four real numbers $0 \leqslant a \leqslant A$ and $0 \leqslant b \leqslant B$ the inequality

$$A + B - 2\sqrt{ab} \geqslant \left(\frac{1}{a+A} + \frac{1}{b+B}\right)\sqrt{(A^2-a^2)(B^2-b^2)}$$

holds.

Proof.

Substituting $A = x(1+\alpha)$, $a = x(1-\alpha)$ $\quad B = y(1+\beta)$, $b = y(1-\beta)$ we get

$$x(1+\alpha) + y(1+\beta) \geqslant 2\sqrt{xy(1-\alpha)(1-\beta)} + 2(x+y)\sqrt{\alpha\beta}$$

to prove. But

$$x(1-\beta) + y(1-\alpha) \geqslant 2\sqrt{xy(1-\alpha)(1-\beta)},$$
$$(x+y)(\alpha+\beta) \geqslant 2(x+y)\sqrt{\alpha\beta}$$

and it remains only to add these inequalities to get the statement desired.

Lemma 2.

Let $f, g : \mathbb{T} \to \mathbb{C}$. Suppose that f is even and g is odd, i.e. $f(-z) = f(z)$, $g(-z) = -g(z)$ for every $z \in \mathbb{T}$. For every set $H \subset \mathbb{T}$ put $F(H) \stackrel{\text{def}}{=} \int_H |f|^2 d\mu$, $G(H) \stackrel{\text{def}}{=} \int_H |g|^2 d\mu$. For every set $E \subset \mathbb{T}$ put $E^+ \stackrel{\text{def}}{=} E \cup (-E)$, $E^- \stackrel{\text{def}}{=} E \cap (-E)$. Then for the function $p = f + g$ one has

$$\int_E |p|^2 d\mu \geqslant \sqrt{\frac{F(E^+)}{F(\mathbb{T})}\frac{F(E^-)}{F(\mathbb{T})}\frac{G(E^+)}{G(\mathbb{T})}\frac{G(E^-)}{G(\mathbb{T})}} \int_{\mathbb{T}} |p|^2 d\mu.$$

Proof.

Let E^* denote the "antisymmetric" part of E, i.e. $E^* \stackrel{\text{def}}{=} \{z \in E : -z \notin E\}$. As $|f|^2$ and $|g|^2$ are both even, we get $F(E^{\pm}) = F(E) \pm F(E^*)$; $G(E^{\pm}) = G(E) \pm F(E^*)$ and therefore

$$\sqrt{\frac{F(E^+)}{F(\mathbb{T})}\frac{F(E^-)}{F(\mathbb{T})}\frac{G(E^+)}{G(\mathbb{T})}\frac{G(E^-)}{G(\mathbb{T})}} \\ = \frac{1}{F(\mathbb{T})G(\mathbb{T})}\sqrt{(F(E)^2 - F(E^*)^2)(G(E)^2 - G(E^*)^2)}$$

Beside it, we have

$$\int_E |p|^2 d\mu = F(E) + G(E) + 2Re \int_E f\bar{g}\, d\mu \\ = F(E) + G(E) + 2Re \int_{E^*} f\bar{g}\, d\mu \geqslant F(E) + G(E) - 2\sqrt{F(E^*)G(E^*)}$$

for $f\bar{g}$ is an odd function and therefore its integral over the symmetric part of E vanishes. At last, $\int_{\mathbb{T}} |p|^2 d\mu = F(\mathbb{T}) + G(\mathbb{T})$. Thus, it is enough to show that

$$F(E) + G(E) - 2\sqrt{F(E^*)G(E^*)} \\ \geqslant \left(\frac{1}{F(\mathbb{T})} + \frac{1}{G(\mathbb{T})}\right)\sqrt{(F(E)^2 - F(E^*)^2)(G(E)^2 - G(E^*)^2)}.$$

But $F(\mathbb{T}) \geqslant F(E^+) = F(E) + F(E^*)$ and $G(\mathbb{T}) \geqslant G(E^+) = G(E) + G(E^*)$. Substituting these trivial estimates into denominators, we see that lemma 2 is a direct consequence of lemma 1 with $A = F(E)$, $a = F(E^*)$, $B = G(E)$ and $b = G(E^*)$.

Let's now prove the inequality

$$\int_E |p|^2 d\mu \geqslant e^{-An\mu(\overline{E})} \int_{\mathbb{T}} |p|^2 d\mu$$

in induction in n (as above we assume that p is a trigonometric polynomial in n characters and $\mu(E) \geqslant \frac{1}{3}$). Note that the estimate (*) for $q = 2$ is equivalent to

$$\int_E |p|^2 d\mu \geqslant \mu(E) e^{-2A(n-1)\mu(\overline{E})} \int_{\mathbb{T}} |p|^2 d\mu$$

which is always weaker except the trivial case $n = 1$.

Inductional base.

If $n = 1$ we have

$$\int_E |p|^2 d\mu = \mu(E) \int_{\mathbb{T}} |p|^2 d\mu \geqslant e^{-An\mu(\overline{E})} \int_{\mathbb{T}} |p|^2 d\mu$$

for every set E of measure $\mu(E) \geqslant \frac{1}{3}$ merely because $t \geqslant e^{-A(1-t)}$ when $t \geqslant \frac{1}{3}$ and $A \geqslant 3$, say.

If $\frac{1}{3} \leqslant \mu(E) \leqslant \frac{2}{3}$, the estimate for "small" sets cited above implies

$$\int_E |p|^2 d\mu \geqslant \mu(E) \Big\{ \frac{\mu(E)}{14e} \Big\}^{2(n-1)} \int_{\mathbb{T}} |p|^2 d\mu$$
$$\geqslant \Big\{ \frac{\mu(E)}{14e} \Big\}^{2n} \int_{\mathbb{T}} |p|^2 d\mu \geqslant e^{-An\mu(\overline{E})} \int_{\mathbb{T}} |p|^2 d\mu$$

for every $n \in \mathbb{N}$ because $\big\{\frac{t}{14e}\big\}^{2n} \geqslant e^{-An(1-t)}$ for $\frac{1}{3} \leqslant t \leqslant \frac{2}{3}$ and $A \geqslant 30$, say.

Inductional step.

Let p be a polynomial in $n > 1$ characters and E be a subset of $\mathbb{T}$ of measure $\mu(E) \geqslant \frac{2}{3}$. Decompose p into the sum of its even and odd parts $f(z) \overset{\text{def}}{=} \frac{1}{2}(p(z) + p(-z))$ and $g(z) \overset{\text{def}}{=} \frac{1}{2}(g(z) - g(-z))$. Without loss of generality we may suppose that both f and g don't vanish identically (otherwise we could take z^2 as a new variable; it is really possible because the "worst" set E (i.e. the set for which the integral $\int_E |p|^2 d\mu$ is the smallest one) is a level set of $|p|$ and thereby has all the symmetries $|p|$ possesses).

Note that f and g are trigonometric polynomials in n_1 and n_2 characters, say, and that $n_1 + n_2 = n$. In notations of lemma 2, we get

$$\int_E |p|^2 d\mu \geqslant \sqrt{\frac{F(E^+)}{F(\mathbb{T})} \frac{F(E^-)}{F(\mathbb{T})} \frac{G(E^+)}{G(\mathbb{T})} \frac{G(E^-)}{G(\mathbb{T})}} \int_{\mathbb{T}} |p|^2 d\mu.$$

Note that $n_1, n_2 < n$ while the inequality $\mu(E) \geqslant \frac{2}{3}$ implies $\mu(E^+) \geqslant \mu(E^-) \geqslant \frac{1}{3}$ and therefore we may apply the inductional assumption to each polinomial f and g and each set E^+ and E^-. It gives

$$\sqrt{\frac{F(E^+)}{F(\mathbb{T})} \frac{F(E^-)}{F(\mathbb{T})} \frac{G(E^+)}{G(\mathbb{T})} \frac{G(E^-)}{G(\mathbb{T})}}$$
$$\geqslant \sqrt{e^{-An_1\mu(\overline{E^+})} e^{-An_1\mu(\overline{E^-})} e^{-An_2\mu(\overline{E^+})} e^{-An_2\mu(\overline{E^-})}}$$
$$= \sqrt{e^{-A(n_1+n_2)(\mu(\overline{E^+})+\mu(\overline{E^-}))}} = e^{-An\mu(\overline{E})},$$

proving the statement.

§2. The case $0 \leqslant q < 2$.

The L^2-estimate for small sets and that for large those can be united in one inequality

$$||p||_{L^2(E)} \geqslant \mu(E)^{A(n-1)} ||p||_{L^2(\mathbb{T})}$$

with some (large) absolute constant A (it follows from the reasoning above that $A = 30$ fits).

We are going to show now that

$$||p||_{L^q(E)} \geqslant \mu(E)^{2A(n-1)} ||p||_{L^q(\mathbb{T})}$$

for every $0 < q < 2$ (the case of L^0-norm can be obtained from here by taking limits of both sides when $q \to 0$). For $\mu(E) \geqslant \frac{1}{3}$ this inequality is equivalent to (*) up to the absolute constant involved. To proceed we need the following classical lemma on convex functions:

Lemma 3.

Let $F, G : [0,1] \to [0,+\infty)$ be two continuously differentiable decreasing convex functions satisfying $F(1) = G(1)$, $F(t) \geqslant G(t)$ for every $t \in [0,1]$. Then for every $\beta \in (0,1]$ and $s \in [0,1]$

$$\int_s^1 |F'(t)|^\beta dt \geqslant \int_s^1 |G'(t)|^\beta dt$$

Let now Φ be the decreasing rearrangement of $|p|$. What we need to prove is

$$\int_\delta^1 \Phi(t)^q dt \geqslant (1-\delta)^{2Aq(n-1)+1} \int_0^1 \Phi(t)^q dt$$

or, equivalently,

$$\int_\delta^1 \Phi(t)^q dt \geqslant \frac{(1-\delta)^{2Aq(n-1)+1}}{1-(1-\delta)^{2Aq(n-1)+1}} \int_0^\delta \Phi(t)^q dt \qquad (0<\delta<1)$$

Let $F(s) \stackrel{\text{def}}{=} \int_s^1 \Phi(t)^2 dt$. Obviously, F is a convex decreasing function on $[0,1]$ and as multiplication of the polinomial p by a constant factor changes nothing, we may assume without loss of generality that $\int_{\mathbb{T}} |p|^2 d\mu = 1$, i.e. $F(0) = 1$. The L^2-estimate above gives $F(t) \geqslant (1-t)^{2A(n-1)+1} \stackrel{\text{def}}{=} G(t)$. Using the Holder inequality we get

$$\begin{aligned}\int_0^\delta \Phi(t)^q dt &\leqslant \delta \left\{ \frac{1}{\delta} \int_0^\delta \Phi(t)^2 dt \right\}^{q/2} \\ &= \delta \left\{ \frac{1}{\delta}(1-F(\delta)) \right\}^{q/2} \leqslant \delta \left\{ \frac{1}{\delta}(1-(1-\delta)^{2A(n-1)+1}) \right\}^{q/2}.\end{aligned}$$

From lemma 3 it follows that

$$\begin{aligned}\int_\delta^1 \Phi(t)^q dt = \int_\delta^1 |F'(t)|^{q/2} \geqslant \\ \int_\delta^1 |G'(t)|^{q/2} = \{2A(n-1)+1\}^{q/2} \frac{1}{Aq(n-1)+1}(1-\delta)^{Aq(n-1)+1}.\end{aligned}$$

So, it is enough to show that

$$\begin{aligned}\{2A(n-1)+1\}^{q/2} &\frac{1}{Aq(n-1)+1}(1-\delta)^{Aq(n-1)+1} \\ &\geqslant \frac{(1-\delta)^{2Aq(n-1)+1}}{1-(1-\delta)^{2Aq(n-1)+1}} \delta \left\{ \frac{1}{\delta}(1-(1-\delta)^{2A(n-1)+1}) \right\}^{q/2}.\end{aligned}$$

But

$$\frac{1}{\delta}(1-(1-\delta)^{2A(n-1)+1}) \leqslant 2A(n-1)+1$$

and therefore it is sufficient to check that

$$\frac{1}{Aq(n-1)+1}(1-\delta)^{Aq(n-1)+1} \geqslant \frac{(1-\delta)^{2Aq(n-1)+1}}{1-(1-\delta)^{2Aq(n-1)+1}}\delta,$$

or, what is the same,

$$1-(1-\delta)^{2Aq(n-1)+1} \geqslant (Aq(n-1)+1)\delta(1-\delta)^{Aq(n-1)}.$$

Denoting $r \overset{\text{def}}{=} Aq(n-1)$ for convenience, we get the inequality $1-(1-\delta)^{2r+1} \geqslant (r+1)\delta(1-\delta)^r$ to prove. But the left hand part is equal to

$$\begin{aligned}(2r+1)\int_0^\delta (1-t)^{2r}dt &= (2r+1)\int_0^{\delta/2}\left\{(1-t)^{2r}+(1-\delta+t)^{2r}\right\} \\ &\geqslant (2r+1)2\int_0^{\delta/2}\left\{(1-t)(1-\delta+t)\right\}^r \geqslant (2r+1)\delta(1-\delta)^r\end{aligned}$$

because $(1-t)(1-\delta+t) \geqslant 1-\delta$ for every $t \in [0,\frac{\delta}{2}]$. This is even stronger than that we need and we have done.

§3. Concluding remarks.

First of all it should be noticed that the classical Turan lemma (as well as its generalization for "small" sets remain valid for exponential polynomials $p(t) = \sum_{k=1}^n c_k e^{i\lambda_k t}$ ($c_k \in \mathbb{C}$, $\lambda_k \in \mathbb{R}$), no a priori separation condition being imposed on λ's. So, for every interval $I \subset \mathbb{R}$, every measurable set $E \subset I$ and every $q \in [0,+\infty]$ we have again

$$||p||_{L^q(I)} \leqslant \left\{\frac{A\mu(I)}{\mu(E)}\right\}^{n-1} ||p||_{L^q(E)}$$

But the argumentation above uses the group structure on $\mathbb{T}$ very hard and I have no idea how to get the good estimate for arbitrary exponential polynomials and sets E close to the whole interval I.

The second remark concerns the case $q>2$, $\mu(E) > 1-\frac{\log n}{n}$ remained. In this case I cannot prove as sharp estimates as for the case $0 \leqslant q \leqslant 2$, but I can aswer completely the following particular question which seems to be of its own interest. The question is how to describe the sets of pairs $\Omega = \{(n,\mu)\}$ for which there exists a constant $B>1$ such that for every pair (n,μ) in Ω and for every polynomial p in n characters and set $E \subset \mathbb{T}$ the inequality $\mu(\overline{E}) \leqslant \mu$ implies $||p||_{L^q(I)} \leqslant B||p||_{L^q(E)}$.

The answer is that for $2 < q < +\infty$ one should demand $\mu \leqslant \frac{\text{Const}}{n^{q/2}}$ for all $(n,\mu) \in \Omega$ and for $q=+\infty$ this condition should be replaced by $\mu \leqslant e^{-\text{const}n}$.