Growth of entire functions with sparse spectra.

Let $f(z)=\sum_{k=0}^{\infty}c_kz^{\lambda_k}$ be an entire function ($c_k\in\mathbb{C}$, $0=\lambda_0<\lambda_1<\dots\in\mathbb{Z}$). Assume that $\sum_{k=1}^{\infty}\frac{1}{\lambda_k}<\infty$.

Old conjecture.

There is no curve Γ joining 0 and ∞ such that f is bounded on Γ (unless $f\equiv const$, of course).

So far the only way to attack this problem was to show that for arbitrarily large $R>0$,

$$M_{\Gamma,f}(R)\stackrel{def}{=}\max_{|z|\leqslant R,z\in\Gamma}|f(z)|\approx M_f(R)\stackrel{def}{=}\max_{|z|\leqslant R}|f(z)|$$

(the idea that actually goes back to Littlewood and 1920's). More precisely, the point was to show that

$$(*)\qquad \overline{\lim_{R\to\infty}}\frac{\log M_{\Gamma,f}(R)}{\log M_f(R)}=1.$$

This particular kind of interpretation is due to the fact that the standard symmetry trick allows to reduce the case of an arbitrary curve Γ to the case $\Gamma=\mathbb{R}_+$, which is much easier to handle.

Below we are going to squeeze from this approach everything it can give. The result is the following

Theorem.

(A) If the sequence $\{\lambda_k\}$ is convex (i.e., if $\lambda_{k+1}-\lambda_k\geqslant\lambda_k-\lambda_{k-1}$), then the condition $\sum_{k=1}^{\infty}\frac{1}{\lambda_k}<\infty$ really implies $(*)$. What is more, for every $\varepsilon>0$, the set $\{R>1:\ M_{\Gamma,f}(R)\leqslant M_f(R)^{1-\varepsilon}\}$ has finite logarithmic measure (i.e., the measure of the set of $\log R$ is finite or, which is the same, the integral of $1/R$ over this set converges).

(B) Under no regularity assumptions about the sequence $\{\lambda_k\}$, the weakest condition that still implies $(*)$ is $\sum_{k=1}^{\infty}\frac{\log\log\lambda_k}{\lambda_k}<\infty$ (or, which is the same, $\sum_{k=1}^{\infty}\frac{\log\log k}{\lambda_k}<\infty$).

Proof.

Let $f(z)=\sum_{k=0}^{\infty}c_kz^{\lambda_k}$. Let $\mu(R)\stackrel{def}{=}\max_k(|c_k|R^{\lambda_k})$ be the central term for $f(z)$ when $|z|=R$, and let $j(R)$ be the central index, i.e., the value of k for which $\mu(R)$ is attained.

Lemma 1.

$M_f(R)\leqslant 4\lambda_{j(R)}^2\mu(R)$ for all $R>1$ except, maybe, a set of finite logarithmic measure.

Proof.

Let $I_j = [a_j, b_j]$ be the interval of values of $\log R$ for which $j(R) = j \geqslant 1$. If $\log R \in [a_j, b_j - \frac{1}{\lambda_j^2}]$, then for every $z \in \mathbb{C}$ such that $|z| = R$, we have

$$|f(z)| = \left|\sum_{k=0}^{\infty} c_k z^{\lambda_k}\right| \leqslant \sum_{k=0}^{\infty} |c_k| R^{\lambda_k} = \sum_{k=0}^{j-1} |c_k| R^{\lambda_k} + \sum_{k=j}^{\infty} |c_k| R^{\lambda_k} \stackrel{def}{=} \Sigma_1 + \Sigma_2$$

Since $|c_k| R^{\lambda_k} \leqslant |c_j| R^{\lambda_j}$ for every k, we have $\Sigma_1 \leqslant j\mu(R)$.

For $k \geqslant j$, we have (taking into account that $\log R$ does not lie too close to the right end of I_j) $|c_k|(R\,e^{1/\lambda_j^2})^{\lambda_k} \leqslant |c_j|(R\,e^{1/\lambda_j^2})^{\lambda_j}$ and therefore

$$|c_k| R^{\lambda_k} \leqslant |c_j| R^{\lambda_k}\, e^{-\frac{\lambda_k - \lambda_j}{\lambda_j^2}} \leqslant \mu(R)\, e^{-\frac{k-j}{\lambda_j^2}}.$$

Thus,

$$\Sigma_2 \leqslant \mu(R) \sum_{k=0}^{\infty} \exp(-\frac{k}{\lambda_j^2}) \leqslant 2\lambda_j^2 \mu(R)$$

Finally we get $M_f(R) \leqslant (j + 2\lambda_j^2)\mu(R) \leqslant 4\lambda_j^2 \mu(R)$.

To finish the proof of the lemma, it remains only to note that the set of $R > 1$ for which $j(R) = 0$ is bounded (if f is not constant) and that the measure of the union of the intervals $[b_j - \frac{1}{\lambda_j^2}, b_j]$ is not greater than $\sum_{k=1}^{\infty} \frac{1}{\lambda_j^2} \leqslant \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty$.

Now let us turn to the proof of the theorem. Consider first the case when the sequence $\{\lambda_k\}$ is convex.

Recall the formula for the distance D in $L^2(0,1)$ from the function x^λ to $span(x^{\lambda_0}, \ldots, x^{\lambda_m})$:

$$D = \frac{1}{\sqrt{2\lambda + 1}} \prod_{k=0}^{m} \frac{|\lambda - \lambda_k|}{\lambda + \lambda_k + 1}.$$

A simple consequence of this formula is that for every polynomial $P(x) = ax^\lambda + \sum_{k=0}^{m} a_k x^{\lambda_k}$, the inequality

$$\max_{x \in [0,r]} |P(x)| \geqslant \frac{|a| r^\lambda}{\sqrt{2\lambda + 1}} \prod_{k=0}^{m} \frac{|\lambda - \lambda_k|}{\lambda + \lambda_k + 1}$$

holds.

Lemma 2.

If $j(R) \geqslant 1$ then $M_{\Gamma,f}(R) \geqslant e^{-15j(R)} \lambda_{j(R)}^{-3} \mu(R)$ for every ray $\Gamma = \{t\xi : t > 0\}$.

Proof.

Without loss of generality we may assume that $\Gamma = \mathbb{R}_+$. Also we shall write j instead of $j(R)$ for convenience.

Write $f(z) = \sum_{k=0}^{\infty} c_k z^{\lambda_k} = \sum_{k=0}^{2j} c_k z^{\lambda_k} + \sum_{k>2j} c_k z^{\lambda_k} \stackrel{def}{=} P(z) + Q(z)$.

Let $\omega > 0$. We have

$$M_{\Gamma,f}(R) \geqslant \max_{x\in[0,Re^{-\omega}]} |f(x)| \geqslant \max_{x\in[0,Re^{-\omega}]} |P(x)| - \max_{x\in[0,Re^{-\omega}]} |Q(x)|.$$

According to the consequence of the distance formula mentioned above,

$$\max_{x\in[0,Re^{-\omega}]} |P(x)| \geqslant |c_j|(R\,e^{-\omega})^{\lambda_j} \frac{1}{\sqrt{2\lambda_j+1}} \prod_{\substack{0\leqslant k\leqslant 2j\\ k\neq j}} \frac{|\lambda_j-\lambda_k|}{\lambda_j+\lambda_k+1}.$$

The convexity of the sequence $\{\lambda_k\}$ implies $\lambda_m \leqslant \frac{m}{n}\lambda_n$ for every $m<n$ and thereby $\frac{|\lambda_n-\lambda_m|}{\lambda_n+\lambda_m+1} \geqslant \frac{|m-n|}{m+n+1}$ for every $m,n\geqslant 0$.

Thus

$$\prod_{\substack{0\leqslant k\leqslant 2j\\ k\neq j}} \frac{|\lambda_j-\lambda_k|}{\lambda_j+\lambda_k+1} \geqslant \prod_{\substack{0\leqslant k\leqslant 2j\\ k\neq j}} \frac{|j-k|}{j+k+1} \geqslant \frac{(j!)^2}{(2j+1)^{2j}} \geqslant e^{-5j}, \text{ say.}$$

Together with the trivial estimate $\frac{1}{\sqrt{2\lambda_j+1}} \geqslant \frac{1}{2\lambda_j}$, this yields

$$\max_{x\in[0,Re^{-\omega}]} |P(x)| \geqslant |c_j|(R\,e^{-\omega})^{\lambda_j} \frac{1}{2\lambda_j}\, e^{-5j} \geqslant \frac{1}{\lambda_j}\, e^{-6j}\, e^{-\omega\lambda_j}\,\mu(R).$$

As to $Q(x)$, we have $|c_k|(R\,e^{-\omega})^{\lambda_k} = |c_k|R^{\lambda_k}\,e^{-\omega\lambda_k} \leqslant |c_j|R^{\lambda_j}\,e^{-\omega\lambda_k} = \mu(R)\,e^{-\omega\lambda_k}$. Thus, for $x\in[0,R\,e^{-\omega}]$,

$$|Q(x)| \leqslant \mu(R)\sum_{k>2j} e^{-\omega\lambda_k} \leqslant \mu(R)\frac{1}{1-\,e^{-\omega}}\, e^{-\omega\lambda_{2j}} \leqslant (1+\frac{1}{\omega})\, e^{-2\omega\lambda_j}\,\mu(R)$$

because (again, due to convexity) $\lambda_{2j} \geqslant 2\lambda_j$.

So, to get a nontrivial estimate for $M_{\Gamma,f}(R)$ from below, we should choose $\omega>0$ satisfying

$$\frac{1}{\lambda_j}\, e^{-6j}\, e^{-\omega\lambda_j} \geqslant 2(1+\frac{1}{\omega})\, e^{-2\omega\lambda_j}$$

Simple computation shows that $\omega = \frac{1}{\lambda_j}(8j + 2\log\lambda_j)$ suffices. This yields the estimate

$$M_{\Gamma,f}(R) \geqslant \frac{1}{2}\lambda_j^{-3}\, e^{-14j}\mu(R) \geqslant \lambda_j^{-3}\, e^{-15j}\mu(R)$$

desired.

The symmetry trick allows to derive from Lemma 2 the estimate

$$M_{\Gamma,f}(R) \geqslant e^{-30j(R)}\lambda_{j(R)}^{-6}\mu(R)^2 M_f(R)^{-1}$$

for an arbitrary curve Γ. Together with the estimate from Lemma 1, it yields the inequality

$$M_{\Gamma,f}(R) \geqslant \frac{1}{4}\, e^{-30j(R)}\lambda_{j(R)}^{-8}\mu(R)$$

valid for all $R>1$ except, maybe, a set of finite logarithmic measure.

Assume now that $M_{\Gamma,f}(R) \leqslant M_f(R)^{1-\varepsilon}$ and that R doesn't belong to the exceptional set. Then we have

$$\frac{1}{4}\, e^{-30j(R)}\lambda_{j(R)}^{-8}\mu(R) \leqslant (4\lambda_j(R)^2\mu(R))^{1-\varepsilon}$$

and therefore

$$(**) \qquad \log\mu(R) \leqslant A(\varepsilon)(j(R) + \log\lambda_{j(R)})$$

Let's show now that the set of all $R > 1$ satisfying $(**)$ has finite logarithmic measure. Let $N(\ell)$ be the number of the elements λ_k that are less than 2^ℓ. The condition $\sum_{k=1}^{\infty}\frac{1}{\lambda_k} < \infty$ is equivalent to $\sum_{\ell\geqslant 1}\frac{N(\ell)}{2^\ell} < \infty$. Note that the slope of $\log\mu(R)$ against $\log R$ is $\lambda_{j(R)}$. Therefore the logarithmic measure of the set of "good" (in the sense of Lemma 1) R for which $(**)$ is satisfied, $\lambda_{j(R)} \in [2^{\ell-1}, 2^\ell)$, $j(R) \geqslant 1$ and $\mu(R) > 1$ (the last two conditions are not very essential since they are fulfilled automatically for all sufficiently large R), doesn't exceed $A(\varepsilon)2^{-(\ell-1)}(j(R) + \log\lambda_{j(R)}) \leqslant 2A(\varepsilon)2^{-\ell}(N(\ell)+\ell)$ because $j(R) \leqslant N(\ell)$ and $\log\lambda_{j(R)} \leqslant \ell$.

Thus, the logarithmic measure of the set of "large" R for which $(**)$ holds doesn't exceed $2A(\varepsilon)\sum_{\ell\geqslant 1}\frac{N(\ell)+\ell}{2^\ell} < \infty$, proving the statement (A) of the theorem.

Let's now show that the condition $\sum_{k\geqslant 2}\frac{\log\log\lambda_k}{\lambda_k} < \infty$ implies $(*)$ without any regularity assumptions. We might just repeat the proof of (A) but I'd like to demonstrate a different technique that will yield a little bit stronger result. Namely, we shall show that for all $R > 1$ except, maybe, a set of finite logarithmic measure, one can find $r \in (0, R)$ such that $m_f(r) \stackrel{def}{=} \min_{|z|=r}|f(z)| \geqslant M_f(R)^{1-\varepsilon}$. (From the proof of (A), it follows only that for most R the connected component of the open set $\{z : |f(z)| < M_f(R)^{1-\varepsilon}\}$ containing the origin lies in some disk of radius strictly less then R and therefore the origin can be surrounded by some simple closed Jordan curve on which f is everywhere large but there is no warranty that this curve is a circumference centered at the origin).

The core of the reasoning below is the following

Lemma 3 (factorization of lacunary polynomials).

Let $P(z) = \sum_{k=0}^{m} c_k z^{\lambda_k}$, $0 \leqslant j \leqslant m$, $d_k \stackrel{def}{=} |\lambda_k - \lambda_j|$. Then

$$P(z) = c_j z^{\lambda_j}\prod_{\substack{0\leqslant k\leqslant m\\ k\neq j}}\varphi_k(z)^{-1}$$

where

$$\varphi_k(z) \stackrel{def}{=} \begin{cases} \dfrac{1}{d_k}\displaystyle\sum_{\ell=1}^{D_k}\frac{z}{z-\zeta_{k\ell}} & \text{if} \quad \lambda_k < \lambda_j; \\[2ex] \dfrac{1}{d_k}\Big(D_k - \displaystyle\sum_{\ell=1}^{D_k}\frac{z}{z-\zeta_{k\ell}}\Big) & \text{if} \quad \lambda_k > \lambda_j, \end{cases}$$

and $D_k \leqslant 2d_k$ for every k.

Proof.

If $d_m \leqslant d_0$ put $Q(z) \stackrel{def}{=} z^{-\lambda_0} P(z)$ and $P_1(z) \stackrel{def}{=} \frac{1}{d_0} z^{\lambda_0+1} \frac{d}{dz}(z^{-\lambda_0} P(z))$.

Note that $P_1(z) = \sum_{k=0}^{m} c_k \frac{\lambda_k - \lambda_0}{d_0} z^{\lambda_k}$ and therefore the term $c_0 z^{\lambda_0}$ disappears while the coefficient at z^{λ_j} doesn't change. Besides,

$$\frac{d_0 P_1(z)}{P(z)} = \frac{zQ'(z)}{Q(z)} = \sum_{\ell=1}^{D_0} \frac{z}{z - \zeta_{0\ell}}$$

where $\zeta_{0\ell}$ are the roots of the polynomial $Q(z)$ of degree $D_0 = \lambda_m - \lambda_0 = d_0 + d_m \leqslant 2d_0$.

If $d_m > d_0$ put $Q(z) \stackrel{def}{=} z^{-\lambda_m} P(z)$ and $P_1(z) \stackrel{def}{=} \frac{1}{d_m} z^{\lambda_m+1} \frac{d}{dz}(z^{-\lambda_m} P(z))$.

Everything remains the same except $Q(z)$ is now a polynomial of degree $D_m \leqslant 2d_m$ in $1/z$, not in z, and thereby

$$\frac{d_m P_1(z)}{P(z)} = \frac{zQ'(z)}{Q(z)} = \sum_{\ell=1}^{D_0} \frac{\frac{1}{z}}{\frac{1}{z} - \frac{1}{\zeta_{m\ell}}} = -\sum_{\ell=1}^{D_m} \frac{\zeta_{m\ell}}{z - \zeta_{m\ell}} = D_m - \sum_{\ell=1}^{D_m} \frac{z}{z - \zeta_{m\ell}}$$

where $1/\zeta_{m\ell}$ are the roots of the polynomial $Q(1/z)$.

Constructing $P_2(z)$ from $P_1(z)$ in the same way and so on, we get the representation desired.

Now we want to show that the functions $\varphi_k(z)$ are small on most circumferences centered at the origin. Note that

$$|\varphi_k(z)| \leqslant 2\left(1 + \frac{1}{D_k} \sum_{\ell=1}^{D_k} \frac{|z|}{|z - \zeta_{k\ell}|}\right) \leqslant 2\left(1 + \frac{1}{D_k} \sum_{\ell=1}^{D_k} \frac{|z|}{\big|\,|z| - |\zeta_{k\ell}|\,\big|}\right).$$

Lemma 3.

Let $\varphi(x) = \frac{1}{D} \sum_{\ell=1}^{D} \frac{x}{|x - x_k|}$ $(x, x_k > 0)$. Then for every $A > 6$, say, the logarithmic measure of the set $\{x > 0 : \varphi(x) > A\}$ doesn't exceed $\frac{9 + 4\log D}{A}$.

Proof.

Let $E_\ell \stackrel{def}{=} \{x>0 : |x-x_\ell| \leqslant \frac{3}{A}x_\ell\}$. We have

$$\frac{x}{|x-x_k|} = \frac{x}{|x-x_k|}\chi_{E_\ell} + \frac{x}{|x-x_k|}\chi_{\mathbb{R}_+\setminus E_\ell} \stackrel{def}{=} \eta_\ell(x)+\theta_\ell(x)$$

Note that for every ℓ we have $\theta_\ell(x) \leqslant \frac{A}{3}(1+\frac{3}{A}) = \frac{A}{3}+1 \leqslant \frac{A}{2}$. Thus, $\frac{1}{D}\sum_{\ell=1}^{D}\theta_\ell(x) \leqslant \frac{A}{2}$. Let $F_\ell \stackrel{def}{=} \{x>0 : |x-x_\ell| \leqslant \frac{3}{AD}x_\ell\}$, $F \stackrel{def}{=} \bigcup_\ell F_\ell$. The logarithmic measure of F doesn't exceed $D\log\frac{AD+3}{AD-3} \leqslant D\frac{9}{AD} = \frac{9}{A}$. To finish the proof of Lemma 3 it remains only to note that for every ℓ

$$\int_{\mathbb{R}_+\setminus F}\eta_\ell(x)\frac{dx}{x} \leqslant \int_{\mathbb{R}_+\setminus F_\ell}\eta_\ell(x)\frac{dx}{x} = 2\int_{\frac{3}{AD}}^{\frac{3}{A}}\frac{dx}{x} = 2\log D.$$

It follows from Lemma 3 that for every $A \geqslant 1$, the logarithmic measure of the set

$$\{r : \text{ there exists } z \text{ for which } |z|=r \text{ and } |\varphi_k(z)| > 4(9+4\log D_k)A\}$$

is less than $\frac{1}{A}$. Though weak type estimates are difficult to add, they are easy to multiply and we conclude that every set $S\subset\mathbb{R}_+$ of logarithmic measure $\omega<1$ contains a point r such that

$$\prod_{\substack{0\leqslant k\leqslant m\\ k\neq j}}\varphi_k(z)^{-1} \geqslant \left(\frac{\omega}{4(9+4\log\lambda_m)e}\right)^m$$

(we have used the trivial estimate $D_k \leqslant \lambda_m$).

Now everything is ready to prove that the condition $\sum\frac{\log\log\lambda_k}{\lambda_k}<\infty$ implies $(*)$.

Let $j=j(R)\geqslant 1$. Assume that $2^{\ell-1}\leqslant\lambda_j<2^\ell$. Again we are going to use the decomposition $f(z)=P(z)+Q(z)$ where

$$P(z) \stackrel{def}{=} \sum_{k:\lambda_k<2^{\ell+2}} c_k z^{\lambda_k}, \qquad Q(z) \stackrel{def}{=} \sum_{k:\lambda_k\geqslant 2^{\ell+2}} c_k z^{\lambda_k}.$$

As before, we observe that for every $|z|\leqslant Re^{-\omega}$,

$$|Q(z)| \leqslant (1+\frac{1}{\omega})e^{-2^{\ell+2}\omega}\mu(R)$$

while there exists $r\in[Re^{-2\omega},Re^{-\omega}]$ such that for every $|z|=r$,

$$|P(z)| \geqslant \left(\frac{\omega}{4(9+4\log(2^{\ell+2}))e}\right)^{N(\ell+2)} e^{-2^{\ell+1}\omega}\mu(R) \geqslant \left(\frac{\omega}{1000\ell}\right)^{N(\ell+2)} e^{-2^{\ell+1}\omega}\mu(R).$$

Again, to get a nontrivial result, we must choose $\omega<1$ for which the estimate for $P(z)$ from below is at least twice as large as the estimate for $Q(z)$ from above. Solving the corresponding inequality, we see that it is possible to take

$$\omega = 2^{-\ell}N(\ell+2)\left[\log(1000\ell)+\log\left(\frac{2^\ell}{N(\ell+2)}\right)+\ell+3\right]$$

(since the condition $\sum_{k=2}^{\infty} \frac{\log\log\lambda_k}{\lambda_k} < \infty$ implies that $2^{-\ell}N(\ell)\log\ell \to 0$ as $\ell \to \infty$, we really have $\omega < 1$ for large ℓ).

Proceeding as before, we find that to get the result desired, it is enough to check that each of the series $\sum_\ell 2^{-\ell}N(\ell)\log\ell$ and $\sum_\ell 2^{-\ell}N(\ell)\log(\frac{2^\ell}{N(\ell)})$ converges. But the convergence of the first series is just a reformulation of the condition $\sum_k \frac{\log\log\lambda_k}{\lambda_k} < \infty$ assumed, while the second series can be split into two parts: the sum over ℓ for which $\frac{N(\ell)}{2^\ell} > \frac{1}{\ell^2}$, which is actually the same as the first series, and the sum over ℓ for which $\frac{N(\ell)}{2^\ell} \leqslant \frac{1}{\ell^2}$, which is dominated by $\sum_\ell \frac{2\log\ell}{\ell^2} < \infty$.

It remains only to show that no weaker condition of the same type is sufficient to get $(*)$ without additional regularity assumptions.

Let $\ell \in \mathbb{Z}$, $\ell > 0$. Put

$$\Delta(\ell) = \log\left(\max_{j:\lambda_j\in[2^\ell,2^{\ell+1})} \prod_{\substack{k:\lambda_k\in[2^\ell,2^{\ell+1})\\ k\neq j}} \frac{\lambda_k+\lambda_j}{|\lambda_k-\lambda_j|}\right)$$

If the number of elements $\lambda_k \in [2^\ell, 2^{\ell+1})$ is less than 2, we put $\Delta(\ell) = 0$ by definition (which is in good accordance with the common agreement that the product of elements of the empty set is 1). Otherwise, obviously, $\Delta(\ell) > 0$.

The statement (A) of the theorem can be generalized in the following way without any essential change in the proof:

Refined version of (A).

If $\sum_\ell \frac{\Delta(\ell)}{2^\ell} < \infty$, then $\overline{\lim}_{R\to\infty} \frac{\log M_{\Gamma,f}(R)}{\log M_f(R)} = 1$.

Now we are going to show that if $\sum_\ell \frac{\Delta(\ell)}{2^\ell} = \infty$, then there exists an entire function $f(z) = \sum_k c_k z^{\lambda_k}$ for which $\lim_{R\to\infty} \frac{\log|f(R)|}{\log M_f(R)} = 0$.

Consider all polynomials $P(x) = \sum_{k:\lambda_k\in[2^\ell,2^{\ell+1})} a_k z^{\lambda_k}$ satisfying $|P(x)| \leqslant \max(1, x^{2^{\ell+2}})$. Let A_ℓ be the maximal possible coefficient of such a polynomial, $\widetilde{\Delta}(\ell) \overset{def}{=} \log A_\ell$. Our first aim is to show that

$$\widetilde{\Delta}(\ell) \geqslant \frac{\Delta(\ell)}{2}.$$

Lemma 4.

Assume that a function $F(z)$ is analytic in the strip $0 < \Re z < 1$ and continuous up to the boundary. Let $\zeta, \zeta_1, \ldots, \zeta_m \in [\frac{1}{4}, \frac{1}{2}]$. If $|F(z)| \leqslant 1$ in the whole strip and $F(\zeta_k) = 0$ for $k = 1, \ldots, m$, then

$$|F(\zeta)| \leqslant \left(\prod_{k=1}^{m} \frac{|\zeta-\zeta_k|}{\zeta+\zeta_k}\right)^{\frac{1}{2}}.$$

Proof.

It is clear that

$$|F(\zeta)| \leqslant \prod_k \frac{\sin\frac{\pi}{2}|\zeta-\zeta_k|}{\sin\frac{\pi}{2}(\zeta+\zeta_k)}$$

(the Blaschke product for the strip). Note now that for every $x \in [0,1]$, we have $x \leqslant \sin\frac{\pi}{2}x \leqslant \frac{\pi}{2}x$ and therefore

$$\frac{\sin\frac{\pi}{2}|\zeta-\zeta_k|}{\sin\frac{\pi}{2}(\zeta+\zeta_k)} \leqslant \frac{\pi}{2}\frac{|\zeta-\zeta_k|}{\zeta+\zeta_k} \leqslant \left(\frac{|\zeta-\zeta_k|}{\zeta+\zeta_k}\right)^{\frac{1}{2}}.$$

(The last inequality is due to the fact that $\frac{|\zeta-\zeta_k|}{\zeta+\zeta_k} \leqslant \frac{1}{3}$ while $\frac{\pi}{2} \leqslant \sqrt{3}$).

Now the inequality $\widetilde{\Delta}(\ell) \geqslant \frac{\Delta(\ell)}{2}$ follows by standard duality argument from the fact that for every (signed) measure μ satisfying $\int_0^1 1d|\mu|(x) + \int_1^\infty x^{2^{\ell+2}}d|\mu|(x) \leqslant 1$ and $\int_0^\infty x^{\lambda_k}d\mu(x) = 0$ for $k \neq j$, the function $F(z) \overset{def}{=} \int_0^\infty x^{2^{\ell+2}z}d\mu(x)$ satisfies the conditions of Lemma 4 (with $\zeta = 2^{-(\ell+2)}\lambda_j$ etc.).

So now we have the condition $\sum_\ell \frac{\widetilde{\Delta}(\ell)}{2^\ell} = \infty$. The sequence $\{\widetilde{\Delta}(\ell)\}$ may have very irregular behavior. Our next aim will be to choose a subsequence $\{\ell_k\}$ such that $\widetilde{\Delta}(\ell_{k+1}) \geqslant 2\widetilde{\Delta}(\ell_k)$ for every k, $\ell_{k+1} - \ell_k \to \infty$ as $k \to \infty$, and still $\sum_k \frac{\widetilde{\Delta}(\ell_k)}{2^{\ell_k}} = \infty$.

Note that for every ℓ_0,

$$\sum_{\substack{\ell \geqslant \ell_0 \\ \widetilde{\Delta}(\ell) \leqslant 2\widetilde{\Delta}(\ell_0)}} \frac{\widetilde{\Delta}(\ell)}{2^\ell} \leqslant 4\frac{\widetilde{\Delta}(\ell_0)}{2^{\ell_0}}.$$

Thus we can construct the sequence $\{\ell_k\}$ by induction taking for every step the least possible ℓ_k for which $\widetilde{\Delta}(\ell_k) \geqslant 2\widetilde{\Delta}(\ell_{k-1})$. This sequence will satisfy our first regularity condition. Now it remains to notice that for every $N > 0$, $k_0 > 0$, at least one of the series $\sum_{m\geqslant 0} \frac{\widetilde{\Delta}(\ell_{k_0+Nm+d})}{2^{k_0+Nm+d}}$ $(d = 1, \dots, N)$ diverges and therefore starting from any k_0, we can pull out a finite subsequence $\{\ell_{k_m}\}$ of the sequence $\{\ell_k\}$ such that $\ell_{k_{m+1}} - \ell_{k_m} \geqslant k_{m+1} - k_m \geqslant N$ and $\sum_m \frac{\widetilde{\Delta}(\ell_{k_m})}{2^{k_m}} > 1$, say. Then we may take $N+1$ instead of N and so on. Dropping several first terms from the sequence $\{\ell_k\}$, we may also assume that $\sum_k e^{-\frac{\widetilde{\Delta}(\ell_k)}{2}} \leqslant 1$.

Now everything is ready to construct the "bad" function $f(z)$. Let $\{R_k\}$ be the sequence of positive numbers defined by $R_1 = 1$, $\log R_k - \log R_{k-1} = \frac{1}{2}\frac{\widetilde{\Delta}(\ell_k)-\widetilde{\Delta}(\ell_{k-1})}{2^{\ell_k+1}}$ for $k \geqslant 2$. Note that the condition $\sum_k \frac{\widetilde{\Delta}(\ell_k)}{2^{\ell_k}} = \infty$ implies that $R_k \to \infty$ as $k \to \infty$.

Let $P_k(x)$ be the polynomial at which $\widetilde{\Delta}(\ell_k)$ is attained. Put

$$f(z) = \sum_k e^{-\frac{\widetilde{\Delta}(\ell_k)}{2}} P_k\Big(\frac{z}{R_k}\Big).$$

The convergence of the series can be easily justified if we note that the condition $|P_k(x)| \leqslant \max(1, x^{2^{\ell_k+2}})$ implies (by subharmonicity of $\log|P_k(z)|$) that $|P_k(z)| \leqslant e^{A2^{\ell_k+2}}$ for every $|z| = 1$ with some absolute constant A (which is just the maximal value over the unit circumference of the harmonic continuation of the function

$u(x) = \log^+ x$ from R_+ to $\mathbb{C} \setminus \mathbb{R}_+$). As $P(z)$ has zero of multiplicity 2^{ℓ_k} at the origin, the Schwartz lemma yields that $|P(z)| \leqslant 1$ for every $|z| \leqslant e^{-4A}$, which is enough to get the convergence of the series defining $f(z)$ for every z.

Let $\psi(x)$ be the piecewise linear function that takes the values $\frac{\widetilde{\Delta}(\ell_k)}{2}$ at the points $\log R_k$ and has the slope 2^{ℓ_k+1} between $\log R_{k-1}$ and $\log R_k$. Note that the existence of a coefficient a_j in P_k for which $|a_j| \geqslant e^{\widetilde{\Delta}(\ell_k)}$ implies that the central term of the function $f(z)$

$$\mu(R) \geqslant \left(\frac{R}{R_k}\right)^{\lambda_j} e^{\frac{\widetilde{\Delta}(\ell_k)}{2}} \geqslant \left(\frac{R}{R_k}\right)^{2^{\ell_k+1}} e^{\frac{\widetilde{\Delta}(\ell_k)}{2}}$$

and therefore

$$\log \mu(R) \geqslant \frac{\widetilde{\Delta}(\ell_k)}{2} + 2^{\ell_k+1}(\log R - \log R_k) = \psi(\log R)$$

for every $R \in [R_{k-1}, R_k]$. Thus, $\log \mu(R) \geqslant \psi(\log R)$ for every $R \geqslant R_1$.

On the other hand, since $\sum_k e^{-\frac{\widetilde{\Delta}(\ell_k)}{2}} \leqslant 1$, we have $|f(R)| \leqslant \max_k |P_k(\frac{R}{R_k})|$ and thereby $\log|f(R)| \leqslant \max_k \log|P_k(\frac{R}{R_k})|$. But

$$\log\left|P_k\left(\frac{R}{R_k}\right)\right| \leqslant \begin{cases} 0 & \text{when } R \leqslant R_k; \\ 2^{\ell_k+2}(\log R - \log R_k) & \text{when } R \geqslant R_k. \end{cases}$$

Note now that for all $x \geqslant \log R_k$ the slope of $\psi(x)$ is at least $2^{\ell_{k+1}}$ and therefore $\log|P_k(\frac{R}{R_k})| \leqslant 2^{\ell_k+2-\ell_{k+1}}\psi(\log R)$.

Since the difference $\ell_{k+1} - \ell_k \to \infty$ as $k \to \infty$, and since for every fixed k, we have $\lim_{R\to\infty} \frac{2^{\ell_k+2}(\log R - \log R_k)}{\psi(\log R)} = 0$ (because the slope of $\psi(x)$ tends to infinity as $x \to \infty$), we may conclude that

$$\lim_{R\to\infty} \frac{\log|f(R)|}{\mu(R)} = 0,$$

which is even (a very little bit) stronger than we needed.

Consider now a condition $\sum_k g(\lambda_k) < \infty$ with some decreasing function $g(x)$ for which $xg(x)$ increases. It is quite an easy exercise to show that this condition implies the convergence of the series $\sum \frac{\Delta(\ell)}{2^\ell}$ (which really governs the situation) if and only if $g(x) \geqslant \mathit{const} \log\log x$, finishing the proof of (B).