Summability of large powers of logarithm of classic lacunary series and its simplest consequences.

This paper can be considered as a small appendix to my large article [1] in "Algebra & Analysis". I'm going to discuss here the last result on lacunary series absolute value distribution obtained there, namely, the following

Theorem 1.

Let $\varepsilon > 0$ be arbitrarily small.

Let $\Lambda = \{m_k\}_{k=-\infty}^{\infty} \subset \mathbb{Z}$ satisfy the Zygmund lacunarity condition, i.e.

$$\sup_{r \neq 0} \ \text{card}\ \{(k', k'') : m_{k'} - m_{k''} = r\} \stackrel{\text{def}}{=} R(\Lambda) < +\infty. \qquad (*)$$

Then for any function $f \epsilon L^2(\mathbb{T})$ spectred at Λ and for each subset $E \subset \mathbb{T}$ of positive measure the inequality

$$\|f\|^2_{L^2(\mathbb{T})} \leq \exp\left\{\frac{B(\varepsilon, R(\Lambda))}{\mu(E)^{2+\varepsilon}}\right\} \int_E |f|^2 d\mu$$

holds with some constant $B(\varepsilon, R(\Lambda))$ depending on ε and $R(\Lambda)$ only. As it has already been mentioned in [1], the power $2+\varepsilon$ in the denominator in the exponent cannot be replaced by $1-\varepsilon$ without additional assumptions on Λ even if we restrict ourselves to arcs E instead of arbitrary measurable sets. Nevertheless, it was commonly believed that much stronger inequality must hold for the most classical example of a spectrum Λ, satisfying $(*)$: that with Hadamard gaps. In particular, it was conjectured that any (arbitrarily large) power of $log|f|$ is integrable while the theorem 1 implies the summability of powers less than $\frac{1}{2}$ only. The following modification of theorem 1 allows to prove this conjecture.

Theorem 2.

Under the conditions of theorem 1, suppose in addition that the difference set $D\Lambda \stackrel{\text{def}}{=} m_{k'} - m_{k''} : k' \neq k''\}$ is a $\Lambda(p)$ - system with $p > 2$ (recall that a set $U \subset \mathbb{Z}$ is called a $\Lambda(p)$ - system if every function $g \ \epsilon \ L^2(\mathbb{T})$ spectred at U belongs to $L^p(\mathbb{T})$ as well and $\|g\|_{L^p(\mathbb{T})} \leq C\|g\|_{L^2(\mathbb{T})}$ with some constant $c > 0$ not depending on g).

Then for any function $f \ \epsilon \ L^2(\mathbb{T})$ spectred at Λ and for each subset $E \subset \mathbb{T}$ of positive measure the inequality

$$\|f\|^2_{L^2(\mathbb{T})} \leq \exp\left\{\frac{B}{\mu(E)^{\frac{4}{p}+\varepsilon}}\right\} \int\limits_E |f|^2 d\mu$$

holds with some constant $B = B(\varepsilon, \Lambda) > 0$ not depending on f and E.

Proof:

The following proposition was the crucial point in the proof of the theorem 1.

Main lemma:

Let $\Lambda \subset \mathbb{Z}$ satisfy the Sigmund lacunarity condition, $E \subset \mathbb{T}$, $\mu(E) \leq \frac{1}{2}$, $f \epsilon\, L^2(\mathbb{T})$ and $spec\, f \subset \Lambda$.

If a positive integer n is so large that

$$\frac{4}{\mu(E')^2} \sum_{\substack{i,j \epsilon \mathbb{Z} \\ i \neq j}} |\hat{\chi}_{E'}(m_i - m_j)|^2 \leq n + 1$$

for each subset $E' \subset E$ of measure $\mu(E') \geq \frac{\mu(E)}{2}$, then there exists a set $E_1 \supset E$ such that

1) $\mu(E_1 \backslash E) \geq \dfrac{\mu(E)}{4n}$

2) $\displaystyle\int_{E_1} |f|^2 d\mu \leq \left\{ \frac{An^5}{\mu(E)^2} \right\}^{3n} \int_{E} |f|^2 d\mu$ with some absolute constant $A > 0$.

This lemma was one of the most complicated results in [1] and to outline its full proof, one should rewrite about a half of 60 pages of [1]. So I have to refer the reader to [1] for details. Nonetheless, once having been proved, it is very easy to apply: we are merely to iterate its result until the measure of the next set E_k, say, will get greater than $\frac{1}{2}$ and after that to use the statement of theorem 1. The main problem is to make the parameter n as small as possible at each step, i.e. to get a good estimate of the sum $\sum\limits_{i \neq j} |\hat{\chi}_{E'}(m_i - m_j)|^2$. In [1] the trivial estimate

$$\sum_{i \neq j} |\hat{\chi}_{E'}(m_i - m_j)|^2 = \sum_{r \neq 0} |\hat{\chi}_{E'}(r)|^2 card\,\{(i, y) : m_i - m_j = r\} \leq$$
$$R(\Lambda) \sum_{r \epsilon \mathbb{Z}} |\hat{\chi}_{E'}(r)|^2 = R(\Lambda)\mu(E')$$

was used, which allowed to take $n \sim \frac{1}{\mu(E)}$. If we take into account that only $r \epsilon D\Lambda$ may occur in the last sum and that for any $\Lambda(p)-$system $U \subset \mathbb{Z}$

$$\Big\{ \sum_{r \epsilon U} \big|\hat{\chi}_{E'}(r)\big|^2 \Big\}^{\frac{1}{2}} = \sup \left\{ \Big| \int_{E'} \overline{g} d\mu \Big| \;:\; \|g\|_{L^2(\mathbb{T})} \leq 1, \;\; \text{spec} \;\; g \subset U \right\} \leq$$
$$\leq \mu(E')^{1-\frac{1}{p}} \sup\{\|g\|_{L^p(\mathbb{T})} : \;\|g\|_{L^2(\mathbb{T})} \leq 1, \;\; \text{spec} \;\; g \subset U\} \leq C\mu(E')^{1-\frac{1}{p}},$$

we shall get much better inequality

$$\sum_{i\neq j} |\hat{\chi}_{E'}(m_i - m_j)|^2 \leq R(\Lambda) C^2 \mu(E')^{2-\frac{2}{p}},$$

which makes possible to put $n = \left[\frac{B}{\mu(E)^{2/p}}\right]$ with some $B > 0$ depending on Λ only ($[x]$ stands here for the entire part of a real number x, as usual).

Thus, if we denote by $C(\nu)$ the best possible constant, for which the inequality $\|f\|^2_{L^2(T)} \leq C(\nu) \int\limits_E |f|^2 d\mu$ holds for every function $f \in L^2(\mathbb{T})$ spectred at Λ and for every set $E \subset \mathbb{T}$ of measure $\mu(E) \geq \nu$, and put $\psi(\nu) \stackrel{\text{def}}{=} \log C(\nu)$, $\Delta(\nu) \stackrel{\text{def}}{=} \frac{1}{4B}\nu^{1+\frac{2}{p}}$, we shall get

$$\frac{\psi(\nu) - \psi(\nu + \Delta(\nu))}{\Delta(\nu)} \leq \frac{12B^2}{\nu^{1+\frac{4}{p}}} \log \frac{AB^5}{\nu^{2+\frac{10}{p}}} \leq \frac{B_\varepsilon}{\nu^{1+\frac{4}{p}+\varepsilon}}$$

for each $0 < \nu \leq \frac{1}{2}$. Comparing this inequality with the corresponding differential that $\frac{d\psi}{d\nu}(\nu) \leq \frac{B_\varepsilon}{\nu^{1+\frac{4}{p}+\varepsilon}}$, we can easily conclude that $\psi(\nu) = O(\nu^{-\frac{4}{p}-\varepsilon})$ as it has been promised. It is well-known that a spectrum Λ with Hadamard gaps does actually satisfy the conditions of the theorem 2 with arbitrarily large $p > 0$ and, therefore, we get the estimate

$$\|f\|^2_{L^2(\mathbb{T})} \leq \exp\left\{\frac{B_\varepsilon}{\mu(E)^\varepsilon}\right\} \int\limits_E |f|^2 d\mu$$

with arbitrarily small $\varepsilon > 0$ and some $B_\varepsilon > 0$ depending on ε and Λ only (not on f or E).

Moreover, we can make p dependent on $\mu(E)$ when choosing n and obtain the better inequality.

$$\|f\|^2_{L^2(\mathbb{T})} \leq \exp\left\{B \log^{10} \frac{2}{\mu(E)}\right\} \int_E |f|^2 d\mu.$$

The power 10 isn't, of course, the best possible one, but it cannot be replaced by any number less than 2 even for arcs E.

Another well-known exmaple, underlying the conditions of the theorem 2, is the following. Let p be an even integer great than 2, $\Lambda = \{m_k\}_{k\in\mathbb{Z}}$ be such that there is no nontrivial linear dependence between integers m_k with integer coefficients for which the sum of the absolute values of the coefficients is $2p$ or less (actually this is an example of a spectrum for which the corresponding constant, connecting L^p and L^2-norms of a function spectred at the difference set, is the least possible one among those for spectra with infinitely many elements). Such a spectrum can be rather dense, namely it can contain about $N^{\frac{1}{2p}-\varepsilon}$ elements among first N successive integers (to construct the corresponding example, it is enough to consider a random spectrum $\Lambda = \{n \in \mathbb{N} : \xi_n = 1\}$, where $\xi_n (n = 1, 2, \cdots)$ are independent random variables defined by $P\{\xi_n = 1\} = \delta n^{\frac{1}{2p}-\varepsilon-1}$, $P\{\xi_n = 0\} = 1 - \delta n^{\frac{1}{2p}-\varepsilon-1}$ with $\delta > 0$ sufficiently small). According to the Mandelbrojt-Belov-Miheev theorem, this means that the power $-\frac{4}{p} - \varepsilon$ in the conclusion of the theorem 2 cannot be replaced by $-\frac{1}{2p-1} + \varepsilon$ even for arcs E (at least for $p = 4, 6, \cdots$). The result of theorem 2 may be used in different ways. We shall mention below only a couple of the most immediate consequences; in both of them we suppose Λ to satisfy the conditions of the theorem 2 with some $p > 4$.

Corollary 3.

Let $\bar{a} = \{a_k\}_{k=-\infty}^{\infty}$ be a sequence of complex numbers for which $\sum\limits_{k=-\infty}^{\infty} |a_k|^2 < +\infty$ and $a_k = 0$ for every $k \notin \Lambda$. Let S be the left shift operator in $\ell^2(\mathbb{Z})$ (i.e. $(\mathcal{S}a)_i = a_{i+1},\ i \in \mathbb{Z}$). Then a is not a cyclic vector for $\mathcal{S}$.

Proof:

This is merely a reformulation of the obvious statement that either $a = 0$ or $\int\limits_{\mathbb{T}} \left| \log \left| \sum\limits_{k\in\mathbb{Z}} a_k z^k \right| \right| d\mu(z) < +\infty.$

Corollary 4.

Let $\Lambda \subset \mathbb{Z}_+$ and $f(z) = \sum\limits_{k\in\Lambda} a_k z^k$ be an analytic function in the unit disk $\mathbb{D}$.

(a) if $\sum\limits_{k\in\Lambda} |a_k|^2 < +\infty$ then $f \in H^2(\mathbb{D})$ and has no singular factor in its canonical factorization,

(b) If $\sum\limits_{k\in\Lambda} |a_k|^2 = +\infty$ then f has infinitely many zeros in $\mathbb{D}$, these zeros being everywhere dense near the boundary of $\mathbb{D}$ and "equidistributed" there in the following sense: let

$$\tau(r)^2 \stackrel{\text{def}}{=} \sum_{k\in\Lambda} |a_k|^2\ r^{2k} \quad (0 < r < 1), \quad \nu_r \stackrel{\text{def}}{=} \sum_{\substack{\xi : f(\xi)=0 \\ 0<|\xi|<r}} \log \frac{r}{|\xi|} \delta_\xi$$

(as usual δ_ξ denotes the unit mass concentrated at ξ and zeros are counted with their multiplicity); then $\frac{1}{\log \tau(r)} \nu_r$ weakly converges to the Lebesgue measure μ on the unit circumference when $r \to 1-$.

Remark.

The statement (b) is worth comparing with the well-known result due to Murai claiming that for spectra $\Lambda \subset \mathbb{Z}_+$ with Hadamard gaps the density of zeros near the boundary may be derived from much weaker assumption $\sup\limits_{z\in\mathbb{D}} |f(z)| = +\infty$ (or, what is the same for series with Hadamard gaps, $\sum\limits_{k\in\Lambda} |a_k| = +\infty$). But Murai's proof uses the Hadamard lacunarity condition in its full strength and I don't know whether there is any essentially denser spectrum Λ for which his result remains valid.

Proof:

The part (a) follows immediately from the simple observation that either $f \equiv 0$ or the integrals $\int\limits_{\mathbb{T}} |\log|f(rz)||^{1+\delta}\, d\mu(z)$ are uniformly bounded for $\frac{1}{2} < r < 1$ if $\delta > 0$ is sufficiently small.

To prove the statement (b) it is enough to check that the measures $\eta_r \overset{\text{def}}{=} \frac{1}{\log\tau(r)}\nu_r$ satisfy the following two conditions:

1) for every $R < 1$ $\eta_r(\mathbb{D}_R) \to 0$ when $r \to 1-$ (here $\mathbb{D}_R \overset{\text{def}}{=} \{z : |z| \le R\}$.

2) if $|\xi| < 1$ and $f(\xi) \ne 0$, then $\int\limits_{\mathbf{C}} \frac{1-|\xi|^2}{|z-\xi|^2}\, d\eta_r(z) \to 1$ when $r \to 1-$

(the set of linear combinations of Poisson kernels is dense in $C(\mathbb{T})$).

(1) is obvious because there are only finitely many zeros of f within $\mathbb{D}_R$ and $\tau(r) \to +\infty$ when $r \to 1-$. When checking (2) we may assume without loss of generality that $f(0) \ne 0$ (the division of f by z^m changes nothing). If $r > |\xi|$ we can write the Jensen formula

$$\sum_{\substack{\zeta:|\zeta|<r\\ f(\zeta)=0}} \log\frac{|r^2 - \overline{\xi}\zeta|}{r|\xi-\eta|} = \int_{\mathbb{T}} \log|f(rz)|\frac{r^2-|\xi|^2}{|\xi - rz|^2} d\mu(z) - \log|f(\xi)|.$$

The result of the theorem 2 implies immediately that the right hand part is $\log\tau(r) + O(1)$ when ξ is fixed and $r \to 1-$.

Therefore

$$\int\limits_{\mathbf{D}} \log\frac{|r^2 - \overline{\xi}z|}{r|z-\xi|}\, d\eta_r(z) \to 1 \quad \text{when} \quad r \to 1- .$$

Note now that
$$\log\frac{|r^2-\overline{\xi}z|}{r|z-\xi|}=\frac{1}{2}\log\left(1+\frac{r^2-|\xi|^2}{|\xi-z|^2}\ (1-|\frac{z}{r}|^2)\right).$$
The ratio $\frac{r^2-|\xi|^2}{|\xi-z|^2}$ is uniformly bounded for all $z\in supp\,\eta_r$ (i.e. for z satisfying $f(z)=0$); and the ratio $\left|\frac{z}{r}\right|$ is very close to 1 when r is close to 1 and $|z|$ is close to 1. Thus we can replace $log\frac{|r^2-\overline{\xi}z|}{r|z-\xi|}$ by the equivalent expression $\frac{1-|\xi|^2}{|z-\xi|^2}log\left|\frac{r}{z}\right|$, getting the result desired. Very similar considerations allow also to prove the following

Corollary 5.

Fix any (small) $\delta>0$ denote $T_\delta\overset{\text{def}}{=}\{x+iy: 1-\delta<x<1,\quad y<\delta(1-x)\}$. If f satisfies the conditions of the corollary 4 then $f(\zeta T_\delta)$ is dense in $\mathbb{C}$ for almost all $\zeta\ \in\ \mathbb{T}$.

Literature cited:

[1]

[2]

[3]