Little Level Theorem and Zero-Khinchin Inequality.
(joint paper with Keith Ball).

Let ξ_0 be any random variable with values in $\mathbb{R}$ (everything works in $\mathbb{R}^n$ too without any essential change in the proof). Let

$$\mathcal{F} = \{\sum_{j=1}^{N} a_j\xi_j : N \geqslant 1; \sum_j a_j^2 = 1;\ \xi_j \text{ are independent and equidistributed with } \xi_0\}.$$

For any $\varepsilon > 0$ define ε-concentration of a random variable ξ as

$$C_\varepsilon(\xi) = \sup_{L\in\mathbb{R}} \mathbb{P}\{|\xi - L| < \varepsilon\}.$$

Then for any $\xi \in \mathcal{F}$ and any $\varepsilon > 0$

$$C_\varepsilon(\xi) \leqslant AC_\varepsilon(\xi_0)$$

where $A > 0$ is some absolute constant (when you consider values in $\mathbb{R}^n$, this constant in our approach blows up as $n \to \infty$). Note that $\xi_0 \in \mathcal{F}$ and therefore the result is up to a certain extent unbeatable.

Proof of the little level theorem.
Let $\xi = \sum_j a_j\xi_j \in \mathcal{F}$.

Consider 2 cases:

Trivial case: there exists j_0 such that $a_{j_0} \geqslant \frac{1}{2}$.
Let $\Omega = \Omega_0 \times \Omega_1$ be the decomposition of the probability space such that ξ_{j_0} depends on $\omega_0 \in \Omega_0$ while all the other ξ_j depend on $\omega_1 \in \Omega_1$. Let $\sigma(\omega_1) = \sum_{j\neq j_0} a_j\xi_j$.

For any fixed $\omega_1 \in \Omega_1$ we have

$$\mathbb{P}_{\omega_0}\{|\xi - L| < \varepsilon\} = \mathbb{P}\Big\{\Big|\xi_{j_0} - \frac{L - \sigma(\omega_1)}{a_{j_0}}\Big| < \frac{\varepsilon}{a_{j_0}}\Big\} \leqslant C_{\varepsilon/a_{j_0}}(\xi_0) \leqslant C_{2\varepsilon}(\xi_0) \leqslant 2C_\varepsilon(\xi_0).$$

Therefore

$$\mathbb{P}\{|\xi - L| < \varepsilon\} = \int_{\Omega_1} \mathbb{P}_{\omega_0}\{|\xi - L| < \varepsilon\} d\mathbb{P}_{\omega_1} \leqslant 2C_\varepsilon(\xi_0)$$

as well and we are done.

Interesting case: all $a_j < \frac{1}{2}$.

Key lemma.
Let $p : \mathbb{R} \to [0, +\infty)$; $\|p\|_{L^1} = 1$; $\|p\|_{L^\infty} = M$. Let $\widehat{p}(t) = \int_{\mathbb{R}} p(x)e^{-2\pi itx}dx$ be the Fourier transform of p.

Then for every $s \geqslant 2$

$$\int_{\mathbb{R}} |\widehat{p}(t)|^s dt \leqslant \frac{AM}{\sqrt{s}}.$$

Suppose that this lemma is proved. Let η_j be independent (and independent of ξ_j) random variables uniformly distributed over the interval $(-\varepsilon, \varepsilon)$. Let

$$\xi'_j = \xi_j + \eta_j; \qquad \eta = \sum_j a_j \eta_j; \qquad \xi' = \sum_j a_j \xi'_j = \xi + \eta.$$

Note that for any $L \in \mathbb{R}$

$$\mathbb{P}\{|\xi' - L| < 3\varepsilon\} \geqslant \mathbb{P}\{|\xi - L| < \varepsilon \,\&\, |\eta| < 2\varepsilon\} = \mathbb{P}\{|\xi - L| < \varepsilon\} \cdot \mathbb{P}\{|\eta| < 2\varepsilon\}.$$

But $\mathbb{E}\eta^2 \leqslant \varepsilon^2$ and therefore the second factor is at least $1/2$. Thus

$$\mathbb{P}\{|\xi - L| < \varepsilon\} \leqslant 2\mathbb{P}\{|\xi' - L| < 3\varepsilon\} \leqslant 2C_{3\varepsilon}(\xi') \leqslant 6C_\varepsilon(\xi').$$

So, it remains to prove that $C_\varepsilon(\xi') \leqslant AC_\varepsilon(\xi_0)$.

Note that the density $p(x)$ of the distribution of every ξ'_j is the same (which is completely irrelevant) and satisfies

$$p(x) \geqslant 0; \qquad \|p\|_{L^1} = 1; \qquad \|p\|_{L^\infty} \leqslant \frac{C_\varepsilon(\xi_0)}{2\varepsilon}$$

(which is very important). For the density $P(x)$ of the distribution of ξ' we have the relation

$$\widehat{P}(t) = \prod_j \widehat{p}(a_j t).$$

Using the Holder inequality, we get

$$\|P\|_{L^\infty} \leqslant \|\widehat{P}\|_{L^1} \leqslant \prod_j \Big\{\int_{\mathbb{R}} |\widehat{p}(a_j t)|^{1/a_j^2} dt\Big\}^{a_j^2}.$$

But (according to the key lemma) for every j

$$\int_{\mathbb{R}} |\widehat{p}(a_j t)|^{1/a_j^2} dt = \frac{1}{a_j} \int_{\mathbb{R}} |\widehat{p}(t)|^{1/a_j^2} dt \leqslant \frac{1}{a_j} A \frac{C_\varepsilon(\xi_0)}{2\varepsilon} a_j = A\frac{C_\varepsilon(\xi_0)}{2\varepsilon}.$$

Hence

$$\|P\|_{L^\infty} \leqslant A\frac{C_\varepsilon(\xi_0)}{2\varepsilon}$$

as well and thereby for any $L \in \mathbb{R}$

$$\mathbb{P}\{|\xi' - L| < \varepsilon\} = \int_{L-\varepsilon}^{L+\varepsilon} P(t)dt \leqslant 2\varepsilon A \frac{C_\varepsilon(\xi_0)}{2\varepsilon} = AC_\varepsilon(\xi_0),$$

finishing the proof.

Proof of the key lemma.

Distribution inequality:

Let $\delta \leqslant \frac{1}{2}$. Then

$$\mu\{t \in \mathbb{R} : |\widehat{p}(t)| \geqslant 1 - \delta^2\} \leqslant 64M\delta.$$

(μ is the Lebesgue measure on $\mathbb{R}$).

Proof of the distribution inequality:

Let $p_-(x) = p(-x)$. Consider the function $f = p_- * p$. We clearly have

$$f \geqslant 0; \qquad \|f\|_{L^1} = 1; \qquad \|f\|_{L^\infty} \leqslant M.$$

Also $\widehat{f}(t) = |p(t)|^2$ and therefore $\widehat{f}(t) \geqslant (1-\delta^2)^2 \geqslant 1-2\delta^2$ whenever $|\widehat{p}(t)| \geqslant 1-\delta^2$.

The main advantage of $f(x)$ compared to $p(x)$ is that it is even and therefore

$$\widehat{f}(t) = \int_{\mathbb{R}} f(x)\cos(2\pi tx)dx = \int_{\mathbb{R}} f(x)(1-2\sin^2(\pi tx))dx = 1-2\int_{\mathbb{R}} f(t)\sin^2(\pi tx)\,dx.$$

Assume first that $\delta = \frac{1}{2m}$ where $m \in \mathbb{N}$. Then

$$\widehat{f}(t) \geqslant 1 - 2\delta^2 \iff \int_{\mathbb{R}} f(t)\sin^2(\pi tx)\,dx \leqslant \delta^2 \implies$$
$$\implies \int_{\mathbb{R}} f(t)\sin^2(\pi\frac{t}{2\delta}x)\,dx \leqslant \frac{1}{4} \iff \widehat{f}\big(\frac{t}{2\delta}\big) \geqslant 1 - 2\frac{1}{4} = \frac{1}{2}$$

(we used the inequality $|\sin(my)| \leqslant m|\sin y|$) and thereby

$$\mu\{t \in \mathbb{R} : |\widehat{p}(t)| \geqslant 1 - \delta^2\} \leqslant \mu\{t \in \mathbb{R} : \widehat{f}(t) \geqslant 1 - 2\delta^2\} \leqslant$$
$$2\delta\mu\{t \in \mathbb{R} : \widehat{f}(t) \geqslant \frac{1}{4}\} \leqslant 32\delta\|\widehat{f}\|^2_{L^2} = 32\delta\|f\|^2_{L^2} \leqslant 32M\delta.$$

To get the result for an arbitrary $\delta < \frac{1}{2}$, just note that we always can find a fraction $\frac{1}{2m}$ between δ and $\min\{2\delta, \frac{1}{2}\}$.

Now it is easy to finish the proof of the key lemma. We have

$$\int_{\mathbb{R}} |\widehat{p}(t)|^s dt = \int_{\{t:|\widehat{p}(t)|\geqslant\frac{3}{4}\}} |\widehat{p}(t)|^s dt + \int_{\{t:|\widehat{p}(t)|<\frac{3}{4}\}} |\widehat{p}(t)|^s dt = I_1 + I_2.$$

Due to the distribution inequality,

$$I_1 \leqslant 64M\int_0^{\frac{1}{2}}(1-\delta^2)^s d\delta \leqslant 64M\int_0^\infty e^{-s\delta^2}d\delta = \frac{A'M}{\sqrt{s}}.$$

At last,

$$I_2 \leqslant \int_{\mathbb{R}} |\widehat{p}(t)|^2\Big(\frac{3}{4}\Big)^{s-2}dt = \int_{\mathbb{R}} |p(t)|^2\Big(\frac{3}{4}\Big)^{s-2}dt \leqslant \Big(\frac{3}{4}\Big)^{s-2}M \leqslant \frac{A''M}{\sqrt{s}}$$

for all $s \geqslant 2$, provided that A'' is large enough.

That's all. We worked under minimal assumptions about the density $p(t)$ of the random variable ξ_0 (it could be just any *measure*). If you assume just a little bit more, namely, that

$$p \in L^{1+\gamma} \quad \text{with some } \gamma > 0,$$

everything gets completely trivial because then for any $\xi = \sum_j a_j \xi_j \in \mathcal{F}$ such that $\max_j a_j \leqslant a(\xi_0)$ and for all $\varepsilon > 0$ one has

$$C_\varepsilon(\xi) \leqslant A(\xi_0)\varepsilon$$

where the constants $a(\xi_0)$ and $A(\xi_0)$ depend a little bit on the sample random variable ξ_0, but not on ε. So, any singularity in the distribution smoothes out very fast.

At last it is worth mentioning that if the Fourier transform of the density of a random variable ξ_0 is positive, then always

$$C_\varepsilon(\xi_0) \leqslant 2\mathbb{P}\{|\xi_0| \leqslant 2\varepsilon\}$$

and therefore under this assumption ξ_0 is the worst guy in the family $\mathcal{F}$. In particular, if $\mathbb{E}\xi_0 = 0$, $\mathbb{E}\xi_0^2 < \infty$, then

$$\|\xi\|_0 \asymp \|\xi\|_2 \quad \forall \xi \in \mathcal{F} \Longleftrightarrow \|\xi_0\|_0 > 0.$$

To get a typical example of such a random variable, just subtract independently any distribution from itself.

Zero-Khinchin inequality:

Let ξ_0 be any random variable with values in $\mathbb{R}$ (everything works in $\mathbb{R}^n$ too without any essential change in the proof) satisfying $\mathbb{E}\xi_0 = 0$ and $\mathbb{E}\xi_0^2 = 1$. Let again

$$\mathcal{F} = \Big\{\sum_{j=1}^{N} a_j \xi_j : N \geqslant 1; \sum_j a_j^2 = 1;\ \xi_j \text{ are independent and equidistributed with } \xi_0\Big\}$$

Then

$$\inf\{\|\xi\|_0 : \xi \in \mathcal{F}\} > 0 \Longleftrightarrow \|\xi_0\|_0 > 0 \,\&\, \|\xi_0 - \xi_0'\|_0 > 0$$

(we will always denote by ξ' a random variable equidistributed with ξ and independent of it).

Proof of the Zero-Khinchin inequality.

$\Longrightarrow$ is trivial. Let us prove $\Longleftarrow$.

Let

$$L(x) = \begin{cases} \log \dfrac{1}{|x|}, & \text{if } |x| < 1; \\ \quad 0 & \text{otherwise} \end{cases} \qquad (x \in \mathbb{R}).$$

We need to show that

$$\sup_{\xi \in \mathcal{F}} \mathbb{E}L(\xi) < +\infty.$$

Key observation:

The Fourier transform $\widehat{L}(t) \geqslant 0$ everywhere and therefore for any two independent random variables ξ and η

$$\mathbb{E}L(\xi-\eta) \leqslant \{\mathbb{E}L(\xi-\xi')\}^{\frac{1}{2}}\{\mathbb{E}L(\eta-\eta')\}^{\frac{1}{2}}.$$

Step 1:
For every $\xi \in \mathcal{F}$

$$\mathbb{E}L(\xi-\xi') \leqslant A\mathbb{E}L(\xi_0-\xi_0').$$

Indeed, let $\xi=\sum_j a_j\xi_j$. Let $\eta_0=\xi_0-\xi_0'$, then $\eta=\xi-\xi'=\sum_j a_j\eta_j$ and, according to Little Level Theorem,

$$\mathbb{E}L(\eta) = \int_0^1 \frac{\mathbb{P}\{|\eta|<\varepsilon\}}{\varepsilon}d\varepsilon \leqslant \int_0^1 \frac{C_\varepsilon(\eta)}{\varepsilon}d\varepsilon$$
$$\leqslant A\int_0^1 \frac{C_\varepsilon(\eta_0)}{\varepsilon}d\varepsilon \leqslant 4A\int_0^1 \frac{\mathbb{P}\{|\eta_0|<\varepsilon\}}{\varepsilon}d\varepsilon = 4A\mathbb{E}L(\eta_0).$$

Step 2:
Suppose that $\xi=\sum_j a_j\xi_j$ and that all $|a_j| \leqslant \frac{1}{\sqrt{2}}$. Then $\xi=\alpha\eta-\beta\zeta$ where $\eta,\zeta\in\mathcal{F}$; $\alpha,\beta\in[\frac{1}{2},1]$, $\alpha^2+\beta^2=1$ and we immediately get

$$\mathbb{E}L(\xi)=\mathbb{E}L(\alpha\eta-\beta\zeta) \leqslant \{\mathbb{E}L(\alpha(\eta-\eta'))\}^{\frac{1}{2}}\{\mathbb{E}L(\beta(\zeta-\zeta'))\}^{\frac{1}{2}}.$$

It remains to note that

$$\mathbb{E}L(\alpha(\eta-\eta')) \leqslant \log\frac{1}{\alpha}+\mathbb{E}L(\eta-\eta') \leqslant \log 2 + A\mathbb{E}L(\xi_0-\xi_0').$$

So, for vast majority of random variables in $\mathcal{F}$ only the condition $\|\xi_0-\xi_0'\|_0>0$ matters.

Step 3:
The rest are "small perturbations" of ξ_0. Forgetting about a non-essential constant factor close to 1, we see that our task is to get an upper bound for $\mathbb{E}L(\xi_0-a\eta)$ where $\eta\in\mathcal{F}$, $a\in(0,1)$.

Split the probability space Ω into 3 parts:

$$\Omega_1=\{\omega:\xi_0(\omega)\leqslant\sqrt{a}\};\qquad \Omega_2=\{\omega:\eta(\omega)\geqslant\frac{1}{2\sqrt{a}}\};\qquad \Omega_3=\Omega\setminus(\Omega_1\cup\Omega_2).$$

Note that for every $\omega\in\Omega_3$

$$|\xi_0(\omega)-a\eta(\omega)| \geqslant \frac{\xi_0}{2}$$

and thereby

$$\int_{\Omega_3} L(\xi_0(\omega)-a\eta(\omega)) \leqslant \mathbb{E}L(\frac{\xi_0}{2}) \leqslant \log 2+\mathbb{E}L(\xi_0).$$

Note also that everywhere

$$L(\xi_0 - a\eta) \leqslant \log \frac{1}{a} + L(a^{-1}\xi_0 - \eta).$$

Since

$$\mathbb{E}L(a^{-1}\xi_0-\eta) \leqslant \left\{\mathbb{E}L(a^{-1}(\xi_0-\xi_0'))\right\}^{\frac{1}{2}}\left\{\mathbb{E}L(\eta-\eta')\right\}^{\frac{1}{2}} \leqslant \left\{\mathbb{E}L(\xi_0-\xi_0')\right\}^{\frac{1}{2}}\left\{\mathbb{E}L(\eta-\eta')\right\}^{\frac{1}{2}},$$

it remains to check that $(\log \frac{1}{a})\mathbb{P}\{\Omega_1 \cup \Omega_2\}$ is bounded by some constant not depending on a.

But

$$\mathbb{P}\{\Omega_1\} \leqslant 2\frac{\mathbb{E}L(\xi_0)}{\log \frac{1}{a}}$$

while

$$\mathbb{P}\{\Omega_2\} \leqslant 4a\mathbb{E}\eta^2 = 4a$$

and we are done.