Ball, Haagerup, and distribution functions

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Abstract. We discuss an elementary trick that sometimes yields simple proofs of integral inequalities of the kind $\int_{\mathcal{X}}(f^s - g^s)\,d\mu \geqslant 0$. We use this trick to obtain "computation-free" proofs of two famous theorems: Ball's theorem on the sections of a cube and Haagerup's theorem on the sharp constants in the Khinchin inequality for Rademacher functions.

Introduction

This paper does not contain any new results. Its only aim is to acquaint the reader with a completely elementary trick that often allows one to prove the integral inequality $\Delta(s) = \int_{X}(f^s - g^s)\,d\mu \geqslant 0$ for $s > s_0$, if one knows that it holds for $s = s_0$ (in particular, if $f, g \in L^s(\mathcal{X}, \mu)$, we are interested in the preservation of the inequality between the L^s-norms of the functions f and g as s increases). Such estimates have various applications in analysis, especially in Banach space geometry.

In this paper we will use our trick to get rid of ugly numerical computations in the proofs of two famous results. The first chapter is devoted to the beautiful theorem of Keith Ball [1] that claims that the maximal $(n-1)$-dimensional volume of a section of the n-dimensional unit cube by a hyperplane is $\sqrt{2}$, i.e., that the maximal section is just the product of the diagonal of the two-dimensional unit square and the $(n-2)$-dimensional unit cube. The second chapter deals with the problem of finding the sharp constants in the Khinchin inequality for Rademacher functions, which was first solved by Uffe Haagerup [5]. We believe that in both cases [1] our proofs are much simpler (both technically and ideologically), shorter, and more elegant than the original ones.

Our main idea will be to use distribution functions, the tool that proved extremely helpful in the probability theory. In this way we shall present a simple sufficient condition for "almost monotonicity" of the integral $\Delta(s)$. More precisely, our condition will allow to conclude that the product $s^{-1}Q^s\Delta(s)$ is increasing for some $Q > 0$.

[1]To be honest, we will prove only one half of Haagerup's theorem in this paper. The second half can be obtained by our methods as well, but the proof becomes more complicated and less transparent. So, after hesitating for a while, we finally decided not to include it.

Chapter I. Ball's theorem on the sections of the unit cube.

Let $t \in \mathbb{R}$. Let $a = (a_1, \dots, a_n)$ be a unit vector in $\mathbb{R}^n$ $(n > 1)$. By $\sigma(a,t)$ we will denote the $(n-1)$-dimensional volume of the section of the cube $\left[-\frac{1}{2}, \frac{1}{2}\right]^n$ by the hyperplane $\{x \in \mathbb{R}^n \, : \, (x,a) = t\}$.

Theorem (Ball). *For any $t \in \mathbb{R}$ and for any unit vector $a \in \mathbb{R}^n$, one has*

$$\sigma(a,t) \leqslant \sqrt{2}.$$

Equality is attained only if $t = 0$ and the vector a has only two non-zero coordinates, each having absolute value $\frac{1}{\sqrt{2}}$, i.e., when the section is a product of the cube $\left[-\frac{1}{2}, \frac{1}{2}\right]^{n-2}$ and the diagonal of the square $\left[-\frac{1}{2}, \frac{1}{2}\right]^2$.

Among other things, this theorem implies that in Euclidean space of sufficiently large dimension all central sections of the ball of n-dimensional volume 1 have greater $(n-1)$-dimensional volumes than the corresponding sections of the unit cube. To prove this (rather counter-intuitive) fact, it is enough to observe that the $(n-1)$-dimensional volumes of the central sections of the ball of n-dimensional volume 1 tend to $\sqrt{e}$ as $n \to +\infty$. Actually, this phenomenon occurs starting from $n = 10$ (see [2] for details).

Thus, Ball's theorem provides the simplest counterexample to the famous conjecture of Busemann and Petty [3]. (They conjectured that of two symmetric with respect to the origin convex bodies in $\mathbb{R}^n$, the one with greater $(n-1)$-dimensional volumes of central sections always has greater n-dimensional volume). The statement of the conjecture is obvious in $\mathbb{R}^2$. Recently it has been proved for $\mathbb{R}^3$ (see [4]). For $5 \leqslant n \leqslant 9$ there are counterexamples (see [6]), though not quite as simple as a ball and a cube. The case $n = 4$ remains controversial to the best of our knowledge. The latest rumor we heard was that the previously claimed counterexample turned out to be faulty and that a proof of the conjecture was found instead. We leave it to the reader to ferret out the truth.

Proof. Fix the unit vector $a \in \mathbb{R}^n$. If at least one coordinate a_j of the vector a vanishes, we can reduce the dimension of the problem. So it is enough to consider the case when all $a_j \neq 0$. Since the cube $\left[-\frac{1}{2}, \frac{1}{2}\right]^n$ is symmetric with respect to each coordinate hyperplane, we may also assume below that $a_j > 0$ for each j.

We shall split the proof of Ball's theorem into several steps.

Step 1. One of the coordinates a_j is large.

Observe that $\sigma(a,t) \leqslant \frac{1}{a_j}$ for each $j = 1, \dots, n$. Indeed, the volume of a section of the unit cube by a hyperplane orthogonal to a equals the volume of its projection to the coordinate hyperplane orthogonal to the j-th ort divided by the cosine of the angle between the vector a and that ort, i.e., by a_j. It remains to note that the projection of *the whole unit cube* to any of the coordinate hyperplanes has volume 1. Thus $\sigma(a,t) < \sqrt{2}$ if one of the coordinates a_j is greater than $\frac{1}{\sqrt{2}}$.

From now on, we shall assume that $a_j \leqslant \frac{1}{\sqrt{2}}$ for all $j = 1, \dots, n$.

Step 2. Integral representation for $\sigma(a,t)$.
Consider the function $S(t) := \sigma(a,t)$. For every $u \in \mathbb{R}$, we have

$$\widehat{S}(u) = \int_{\mathbb{R}} \sigma(a,t) e^{-2\pi i u t} dt = \int_{[-\frac{1}{2},\frac{1}{2}]^n} e^{-2\pi i u (x,a)}\, dx$$
$$= \prod_{j=1}^{n} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i u a_j x_j} dx_j = \prod_{j=1}^{n} \frac{\sin \pi a_j u}{\pi a_j u}.$$

Taking the inverse Fourier transform, we get

$$\sigma(a,t) = S(t) = \int_{-\infty}^{\infty} \widehat{S}(u) e^{2\pi i u t} du = \int_{-\infty}^{\infty} e^{2\pi i u t} \prod_{j=1}^{n} \frac{\sin \pi a_j u}{\pi a_j u} du.$$

From this representation one can derive that the naive conjecture that the section of the largest volume is the central section orthogonal to the main diagonal is false. Indeed, the volume of this section (corresponding to $t = 0$, $a_j = \frac{1}{\sqrt{n}}$) equals

$$\int_{\mathbb{R}} \left(\frac{\sin \frac{\pi u}{\sqrt{n}}}{\frac{\pi u}{\sqrt{n}}} \right)^n du = \sqrt{n} \int_{\mathbb{R}} \left(\frac{\sin \pi v}{\pi v} \right)^n dv$$

Using the Laplas asymptotic formula, we find that this volume tends to $\sqrt{\frac{6}{\pi}} < \sqrt{2}$ as $n \to \infty$.

Step 3. Reduction to Ball's integral inequality.
Since $a_1^2 + \cdots + a_n^2 = 1$, the integral representation of $\sigma(a,t)$ together with the Hölder inequality imply

$$\sigma(a,t) \leqslant \int_{\mathbb{R}} \prod_{j=1}^{n} \left| \frac{\sin \pi a_j u}{\pi a_j u} \right| du \leqslant \prod_{j=1}^{n} \left(\int_{\mathbb{R}} \left| \frac{\sin \pi a_j u}{\pi a_j u} \right|^{1/a_j^2} du \right)^{a_j^2}.$$

Thus, we need only show that every integral $\displaystyle\int_{\mathbb{R}} \left| \frac{\sin \pi a_j u}{\pi a_j u} \right|^{1/a_j^2} du$ does not exceed $\sqrt{2}$ and that equality is possible only when $a_j = \frac{1}{\sqrt{2}}$. Taking into account that $\frac{1}{a_j^2} \geqslant 2$ for every j and making the change of variable $u \to a_j u$, we see that it would suffice to prove the following **Ball's integral inequality**:

$$\int_{\mathbb{R}} \left| \frac{\sin \pi x}{\pi x} \right|^s dx < \sqrt{\frac{2}{s}} \qquad \text{for every } s > 2$$

(for $s = 2$, it turns into identity).

Ball's original proof of this inequality, though quite straightforward, involved a lot of unpalatable numerical computations some of which were almost impossible to verify without a calculator. Below we shall use a completely different approach.

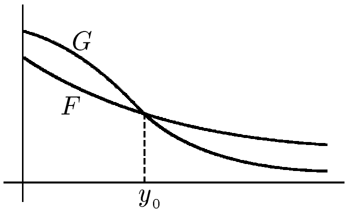
Step 4. Lemma on distribution functions.

Let $(\mathcal{X}, \mu)$ be a measure space. For a non-negative function $f : \mathcal{X} \to [0, +\infty)$, its **distribution function** $F(y)$, $y > 0$ is defined by

$$F(y) := \mu\{x \in \mathcal{X} \,:\, f(x) > y\}.$$

Clearly, F is a non-increasing function on $(0, +\infty)$.

Lemma on distribution functions. *Let f and g be any two nonnegative measurable functions on a measure space $(\mathcal{X}, \mu)$. Let F and G be their distribution functions. Assume that both $F(y)$ and $G(y)$ are finite for every $y > 0$. Assume also that at some point y_0 the difference $F - G$ changes sign from $-$ to $+$, i.e., $F(y) \leqslant G(y)$ for all $y \in (0, y_0)$ and $F(y) \geqslant G(y)$ for all $y > y_0$.*



Let $S = \{s > 0 \,:\, f^s - g^s \in L^1(\mathcal{X}, \mu)\}$. Then the function

$$\varphi(s) = \frac{1}{s y_0^s} \int_{\mathcal{X}} (f^s - g^s) \; d\mu$$

is increasing on S. In particular, if $\int_{\mathcal{X}} (f^{s_0} - g^{s_0}) d\mu = 0$, then $\int_{\mathcal{X}} (f^s - g^s) d\mu \geqslant 0$ for each $s > s_0$. The equality may hold only if the functions F and G coincide.

Remark. *Note that we do not demand finiteness of the integrals $\int_{\mathcal{X}} f^s \, d\mu$ and $\int_{\mathcal{X}} g^s \, d\mu$ in the definition of the set S. Only the difference $f^s - g^s$ is required to be Lebesgue integrable with respect to the measure μ. Though this observation is of little importance now, it will play crucial role in Chapter II.*

Proof. Observe first of all, that if the difference $f - g$ of two nonnegative functions f and g is Lebesgue integrable and both their distribution functions $F(y)$ and $G(y)$ are finite for all $y > 0$, then

$$\int_{\mathcal{X}} (f - g) \, d\mu = \int_0^{\infty} (F(y) - G(y)) \; dy.$$

Indeed, let $h(x) = \min(f(x), g(x))$. The distribution function $H(y)$ of the function h does not exceed $F(y)$ and therefore is finite for every $y > 0$. Also, the (non-negative) difference $f - h$ does not exceed $|f - g|$, and thereby is Lebesgue integrable. Now, applying the Fubini theorem to the characteristic function of the set $\{(x, y) \in \mathcal{X} \times (0, +\infty) \,:\, h(x) \leqslant y < f(x)\}$, we obtain

$$0 \leqslant \int_{\mathcal{X}} (f - h) \, d\mu = \int_0^{\infty} (F(y) - H(y)) \, dy < +\infty.$$

Analogously,

$$0 \leqslant \int_{\mathcal{X}} (g-h)\,d\mu = \int_0^\infty (G(y)-H(y))\,dy < +\infty.$$

Subtracting the second identity from the first, we obtain the desired statement.

Since for $s>0$ the distribution function of, say, f^s equals $F(y^{1/s})$, we get

$$\int_{\mathcal{X}} (f^s-g^s)\,d\mu = \int_0^\infty \left(F(y^{1/s})-G(y^{1/s})\right)dy = s\int_0^\infty y^{s-1}\,(F(y)-G(y))\,dy.$$

Therefore for $s>s_0$ ($s,s_0\in S$), we have

$$\varphi(s)-\varphi(s_0) = \frac{1}{y_0}\int_0^\infty \left(\left(\frac{y}{y_0}\right)^{s-1} - \left(\frac{y}{y_0}\right)^{s_0-1}\right)(F(y)-G(y))\,dy \geqslant 0,$$

because both factors in the integrand simultaneously change their signs at y_0. $\square$

Step 5. Proof of Ball's integral inequality.

Using the Euler-Poisson integral $\int_{\mathbb{R}} e^{-\pi x^2}dx = 1$, we can rewrite Ball's inequality as $\int_0^\infty g^s < \int_0^\infty f^s$, where $f(x)=e^{-\pi x^2/2}$, $g(x)=\left|\frac{\sin \pi x}{\pi x}\right|$. We need to prove this inequality for all powers $s>2$. Since for $s=2$ we have an identity, it is enough to show that the corresponding distribution functions F and G satisfy the condition of the lemma from Step 4 (with $\mathcal{X}=(0,\infty)$, $d\mu = dx$, $s_0=2$). Observe that both the functions f and g are strictly less than 1 on the whole ray $(0,+\infty)$ and thereby $F(y)=G(y)=0$ for each $y\geqslant 1$. So, we may restrict ourselves to the values $y\in(0,1)$.

Since f is decreasing on $(0,\infty)$, we have

$$F(y) = f^{-1}(y) = \sqrt{\frac{2}{\pi}\ln\frac{1}{y}}.$$

To estimate $G(y)$, consider the quantities $y_m = \max_{[m,m+1]} g$, $m\in\mathbb{N}$ (the graph of $g(x)$ for $x\geqslant 1$ looks as a sequence of decreasing humps and y_m is just the height of the m-th hump in this sequence – see the picture below). Obviously, $\dfrac{1}{\pi(m+\frac{1}{2})} < y_m < \dfrac{1}{\pi m}$.

For $x\in(0,1)$, we have

$$g(x) = \prod_{k=1}^\infty \left(1-\frac{x^2}{k^2}\right) \leqslant \prod_{k=1}^\infty e^{-x^2/k^2} = e^{-(\pi x)^2/6} < e^{-\pi x^2/2} = f(x).$$

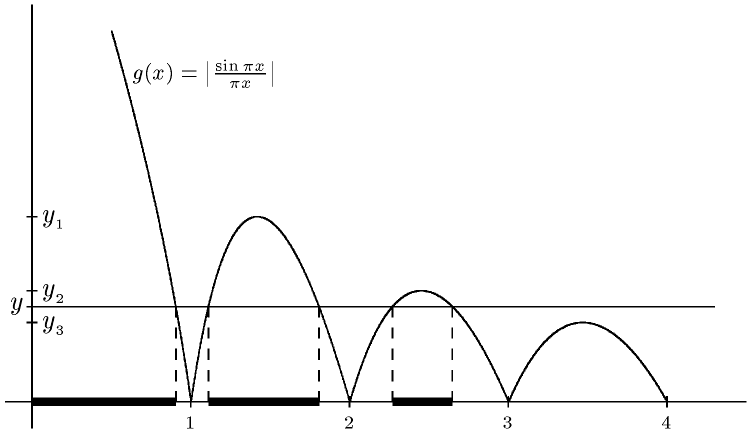
Therefore, for all $y\in(y_1,1)$,

$$G(y) = \mu\{x\in(0,1) \,:\, g(x)>y\} < \mu\{x\in(0,1) \,:\, f(x)>y\} \leqslant F(y),$$

i.e., the difference $F-G$ is positive on the interval $(y_1,1)$.

The difference $F-G$ must change sign *at least* once on $(0,+\infty)$ because $\int_0^\infty 2y(F(y)-G(y))\,dy=\int_0^\infty (f(x)^2-g(x)^2)\,dx=0$. To prove that the change of sign occurs *only* once, it now suffices to show that $F-G$ is increasing on $(0,y_1)$. Since both F and G are continuous decreasing functions, it is enough to prove that $|G'(y)|>|F'(y)|$ on each interval (y_{m+1},y_m), $m\in\mathbb{N}$.

For $y\in(0,y_1)$, the value $G(y)$ can be visualized as the sum of lengths of the thick black intervals on the picture below.



$G(y)$ **equals the total length of the thick black intervals.**

It is clear from this picture that for each $y\in(0,y_1)$, $y\neq y_j$, the function G is differentiable and

$$|G'(y)|=\sum_{x>0\,:\,g(x)=y}\frac{1}{|g'(x)|}.$$

When $y\in(y_{m+1},y_m)$, the equation $g(x)=y$ has one root on the interval $(0,1)$ and two roots on each interval $(k,k+1)$, $k=1,\dots,m$. Let us estimate $|g'(x)|$ from above at these roots.

- If the root $x\in(k,k+1)$, $k\geqslant 1$, then

$$|g'(x)|=\left|\frac{\cos\pi x}{x}-\frac{\sin\pi x}{\pi x^2}\right|\leqslant\frac{1}{x}\left(1+\frac{|\sin\pi(x-k)|}{\pi x}\right)\leqslant\frac{1}{x}\left(1+\frac{\pi(x-k)}{\pi k}\right)=\frac{1}{k};$$

- If the root $x\in(0,1)$, then

$$|g'(x)|=\frac{\sin\pi x-\pi x\cos\pi x}{\pi x^2}=\frac{1}{\pi x^2}\int_0^{\pi x}t\sin t\,dt\leqslant\frac{1}{\pi x^2}\int_0^{\pi x}t\,dt=\frac{\pi}{2}.$$

Thus for every $y\in(y_{m+1},y_m)$, we have

$$|G'(y)|\geqslant\frac{2}{\pi}+2\sum_{k=1}^m k=\frac{2}{\pi}+m+m^2.$$

Recalling that $F(y)=\sqrt{\frac{2}{\pi}\ln\frac{1}{y}}$, we obtain

$$\left|\frac{G'(y)}{F'(y)}\right| = |G'(y)|\, y\sqrt{2\pi\ln\frac{1}{y}} \geqslant \left(\frac{2}{\pi}+m+m^2\right) y\sqrt{2\pi\ln\frac{1}{y}}.$$

Since $y_1<\frac{1}{\pi}<\frac{1}{\sqrt{e}}$, the function $y\sqrt{2\pi\ln\frac{1}{y}}$ is increasing on $(0,y_1)$. Recalling that $y>y_{m+1}>\dfrac{1}{\pi(m+\frac{3}{2})}$, we get

$$\left|\frac{G'(y)}{F'(y)}\right| > \frac{\frac{2}{\pi}+m+m^2}{\frac{3}{2}+m}\sqrt{\frac{2}{\pi}\ln\pi(m+\frac{3}{2})} > \sqrt{\frac{2}{\pi}\ln\frac{5\pi}{2}}$$

($m\geqslant 1$ and $\frac{2}{\pi}+m^2>\frac{1}{2}+1=\frac{3}{2}$).

It remains to note that the right hand part is greater than 1 because $\ln 5x>x$ for every $x\in[1,2]$, and, in particular, for $x=\frac{\pi}{2}$ (since the function $\ln 5x$ is concave, it is enough to check this estimate at the ends of the interval, where it is obvious).

Ball's inequality is thus completely proved. $\square$

Chapter II. Haagerup's theorem on sharp constants in the Khinchin inequality.

Recall that Rademacher functions $r_k(t)$ $(k = 1, 2, \dots,\ t \in [0, 1])$ are defined by

$$r_k(t) = \operatorname{sign} \sin(2^k \pi t)$$

From the probabilistic point of view, they are just independent random variables assuming values $+1$ and -1 with probability $\frac{1}{2}$ each. One of the most famous results on Rademacher functions is the **Khinchin inequality**:

$$A_p \left(\sum_{k=1}^{n} a_k^2 \right)^{p/2} \leqslant R_p(a) := \int_0^1 \left| \sum_{k=1}^{n} a_k r_k(t) \right|^p dt \leqslant B_p \left(\sum_{k=1}^{n} a_k^2 \right)^{p/2}$$

for any real coefficient vector $a = (a_1, \dots, a_n)$ and for any $0 < p < +\infty$. Uffe Haagerup found the sharp values of the constants A_p and B_p in this inequality for all powers $p > 0$ (see [5]).

Formulation of Haagerup's theorem.

To find the best constants A_p and B_p in the Khinchin inequality is the same as to solve the extremal problems

$$\int_0^1 \left| \sum_{k=1}^{n} a_k r_k(t) \right|^p dt \to \min, \qquad \sum_k a_k^2 = 1$$

and

$$\int_0^1 \left| \sum_{k=1}^{n} a_k r_k(t) \right|^p dt \to \max, \qquad \sum_k a_k^2 = 1$$

respectively. There are three "natural" linear combinations $\sum_k a_k r_k(t)$ to look at:

- $n = 1$, $a_1 = 1$ (a single Rademacher function). In this case $R_p(a) = 1$.
- $n = 2$, $a_1 = a_2 = \frac{1}{\sqrt{2}}$ (the sum or the difference of two Rademacher functions). In this case $R_p(a) = 2^{\frac{p-2}{2}}$.
- $n \to +\infty$, $a_1 = \dots = a_n = \frac{1}{\sqrt{n}}$. In this case it is more convenient to accept the probabilistic point of view and to conclude, using the central limit theorem, that we are actually talking about the normal distribution. Hence, the corresponding (limit) value of $R_p(a)$ equals

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |x|^p e^{-\frac{x^2}{2}} dx = \sqrt{\frac{2}{\pi}} \int_0^\infty x^p e^{-\frac{x^2}{2}} dx$$
$$= 2^{\frac{p-1}{2}} \sqrt{\frac{2}{\pi}} \int_0^\infty y^{\frac{p-1}{2}} e^{-y} dy = \frac{2^{\frac{p}{2}}}{\sqrt{\pi}} \Gamma\left(\frac{p+1}{2}\right) = 2^{\frac{p-2}{2}} \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}$$

(the statement of the central limit theorem for bounded symmetric random variables is strong enough to ensure convergence of moments of any order).

Now Haagerup's theorem can be stated in a very simple way:

Theorem (Haagerup). *For each $p \in (0, +\infty)$, the extremum in either of the extremal problems is one of these three values.*

What does this mean in terms of formulae? First of all, note that $A_p = 1$ for $p \geqslant 2$ and $B_p = 1$ for $p \leqslant 2$ (this is just the Hölder inequality). So, the only non-trivial tasks are to find A_p for $p < 2$ and to find B_p for $p > 2$. Haagerup's theorem claims that in these cases the extreme value is delivered by either the sum of two Rademacher functions or by the normal distribution, i.e.,

$$A_p = 2^{\frac{p-2}{2}} \min\left(1, \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{3}{2})}\right); \qquad B_p = 2^{\frac{p-2}{2}} \max\left(1, \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{3}{2})}\right).$$

Using the alternative representation for the Γ-function

$$\Gamma(q) = \int_0^\infty t^q e^{-t}\frac{dt}{t} = \lim_{n\to\infty}\int_0^n t^q\left(1-\frac{t}{n}\right)^n\frac{dt}{t} = \lim_{n\to\infty}\frac{n^q\, n!}{q(q+1)\ldots(q+n)},$$

we conclude that the second derivative $\frac{d^2}{dq^2}\ln\Gamma(q) = \sum\limits_{k=0}^{\infty}\frac{1}{(q+k)^2}$ of the function $\ln\Gamma(q)$ is *positive and decreasing* on $(0, +\infty)$. Since $\Gamma(1) = \Gamma(2) = 1$, we can derive from here that the function $\ln\Gamma(q)$ is convex on $(0, +\infty)$ and attains its minimum on the interval $[1, 2]$ somewhere to the left from its middle point $\frac{3}{2}$. In particular, this implies that the function Γ is increasing on $[\frac{3}{2}, +\infty)$. So for $p > 2$, the maximum of $R_p(a)$ is delivered by the normal distribution and

$$B_p = 2^{\frac{p-2}{2}}\frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{3}{2})}.$$

The case $p < 2$ is slightly more complicated because there exists $q_0 \in (1, \frac{3}{2})$ such that $\Gamma(q) > \Gamma(\frac{3}{2})$ for $0 < q < q_0$, $\Gamma(q_0) = \Gamma(\frac{3}{2})$, and $\Gamma(q) < \Gamma(\frac{3}{2})$ for $q_0 < q < \frac{3}{2}$. Therefore

$$A_p = \begin{cases} 2^{\frac{p-2}{2}}, & \text{if } 0 < p \leqslant p_0; \\ 2^{\frac{p-2}{2}}\frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{3}{2})}, & \text{if } p_0 \leqslant p \leqslant 2 \end{cases}$$

where $p_0 = 2q_0 - 1 \in (1, 2)$ is the only nontrivial positive root of the equation $\Gamma(\frac{p+1}{2}) = \Gamma(\frac{3}{2})$.

Our aim in this chapter will be to give a (relatively) simple proof of one half of Haagerup's theorem. Namely, we shall consider the case $p < 2$ (or, which is quite the same, shall find the sharp value of A_p for all $p > 0$). Haagerup also considers the cases $p < 2$ and $p > 2$ separately and it is almost impossible to decide for which of them his original proof is more painstaking.

Proof. Again, we will split the proof into several steps.

Step 1. Reduction of the case $0 < p < p_0$ to the case $p = p_0$.
Assume that we already know that $A_{p_0} = 2^{\frac{p_0-2}{2}}$, i.e., that $R_p(a) \geqslant 2^{\frac{p_0-2}{2}}$ for any $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ satisfying $\sum_k a_k^2 = 1$. Let now $0 < p < p_0$. Then, using the Hölder inequality

$$R_{p_0}(a) \leqslant R_p(a)^{\frac{2-p_0}{2-p}} \, R_2(a)^{\frac{p_0-p}{2-p}}$$

and taking into account that $R_2(a) = \sum_k a_k^2 = 1$, we conclude that

$$R_p(a) \geqslant \left(2^{\frac{p_0-2}{2}}\right)^{\frac{2-p}{2-p_0}} = 2^{\frac{p-2}{2}},$$

so $A_p = 2^{\frac{p-2}{2}}$ for all $0 < p < p_0$.
Thus we need only consider the powers $p_0 \leqslant p \leqslant 2$.

Step 2. Integral representation for $R_p(a)$.
Let $0 < p < 2$. Then for every $s \in \mathbb{R}$,

$$|s|^p = C_p \int_0^\infty \frac{1 - \cos us}{u^{p+1}} \, du$$

where the constant $C_p > 0$ depends on p only. Plugging $s = \sum_1^n a_k r_k(t)$ into this identity and integrating over $t \in [0,1]$, we find

$$\int_0^1 \left| \sum_1^n a_k r_k(t) \right|^p dt = C_p \int_0^\infty \frac{1 - \int_0^1 \cos\left(u \sum_1^n a_k r_k(t)\right) \, dt}{u^{p+1}} \, du.$$

Observing that

$$\int_0^1 \cos\left(u \textstyle\sum_1^n a_k r_k(t)\right) = \operatorname{Re} \displaystyle\int_0^1 e^{iu \sum_1^n a_k r_k(t)} = \prod_1^n \cos a_k u,$$

we finally arrive at **Haagerup's integral representation**:

$$R_p(a) = \int_0^1 \left| \sum_1^n a_k r_k(t) \right|^p dt = C_p \int_0^\infty \frac{1 - \prod_1^n \cos a_k u}{u^{p+1}} \, du.$$

Step 3. The case of small coefficients: reduction to Haagerup's integral inequality.
Let us again assume that $\sum_j a_k^2 = 1$. Young's inequality yields

$$\prod_1^n \cos a_k u \leqslant \prod_1^n |\cos a_k u| \leqslant \sum_1^n a_k^2 \, |\cos a_k u|^{1/a_k^2} = 1 - \sum_1^n a_k^2 (1 - |\cos a_k u|^{1/a_k^2})$$

and therefore

$$\int_0^1 \left| \sum_1^n a_k r_k(t) \right|^p dt \geqslant \sum_1^n a_k^2 \, \mathcal{I}_p\left(\frac{1}{a_k^2}\right) \qquad (*)$$

where

$$\mathcal{I}_p(s) = C_p \int_0^\infty \left(1 - \left|\cos \frac{u}{\sqrt{s}}\right|^s\right) \frac{du}{u^{p+1}} \qquad (s > 0).$$

The value of the integral $\mathcal{I}_p(s)$ can be computed in two cases:

- **Case 1:** $s = 2$. It follows immediately from Haagerup's integral representation for $n = 2$, $a_1 = a_2 = \frac{1}{\sqrt{2}}$ that
$$\mathcal{I}_p(2) = 2^{\frac{p-2}{2}}.$$

- **Case 2:** $s = \infty$. The trivial estimate $\ln\cos\theta = O(\theta^2)$ $(\theta \to 0)$ implies that $1 - \left|\cos\frac{u}{\sqrt{s}}\right|^s \leqslant 1 - e^{-\mathrm{Const}\, u^2} \leqslant \mathrm{Const}\, u^2$ for $0 < u < 1$ and sufficiently large s. Since this difference also never exceeds 1, we may apply Lebesgue's dominated convergence theorem and conclude that the limit $\lim_{s\to\infty} \mathcal{I}_p(s)$ does exist and equals
$$\mathcal{I}_p(\infty) = C_p \int_0^\infty \left(1 - e^{-\frac{u^2}{2}}\right) \frac{du}{u^{p+1}}.$$
On the other hand, for every *even* $n \in \mathbb{N}$, Haagerup's integral representation with $a_1 = \cdots = a_n = \frac{1}{\sqrt{n}}$ yields
$$R_p(\tfrac{1}{\sqrt{n}}, \ldots, \tfrac{1}{\sqrt{n}}) = \mathcal{I}_p(n).$$
As we have seen above, the left hand part tends to $2^{\frac{p-2}{2}} \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{3}{2})}$ (which is our hypothetical value for A_p) as $n \to \infty$. Thus we also have the identity
$$\mathcal{I}_p(\infty) = 2^{\frac{p-2}{2}} \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{3}{2})}.$$

Recall that $\mathcal{I}_p(2) \geqslant \mathcal{I}_p(\infty)$ for all $p \in [p_0, 2)$. If we could only show that $\mathcal{I}_p(s) \geqslant \mathcal{I}_p(\infty)$ for all other $s \in [1, +\infty)$ as well, then the inequality $(*)$ would yield
$$\int_0^1 \left|\sum_1^n a_k r_k(t)\right|^p dt \geqslant \left(\sum_1^n a_k^2\right) \mathcal{I}_p(\infty) = \mathcal{I}_p(\infty) = 2^{\frac{p-2}{2}} \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)},$$
proving Haagerup's theorem. Note now that
$$\mathcal{I}_p(s) - \mathcal{I}_p(\infty) = C_p \int_0^\infty \left(1 - \left|\cos\frac{u}{\sqrt{s}}\right|^s\right) \frac{du}{u^{p+1}} - C_p \int_0^\infty \left(1 - e^{-\frac{u^2}{2}}\right) \frac{du}{u^{p+1}}$$
$$\stackrel{(u = x\sqrt{s})}{=} C_p s^{\frac{p}{2}} \int_0^\infty \left(e^{-s\frac{x^2}{2}} - |\cos x|^s\right) \frac{dx}{x^{p+1}}.$$
So, the desired estimate $\mathcal{I}_p(s) \geqslant \mathcal{I}_p(\infty)$ is equivalent to **Haagerup's integral inequality**
$$H(p, s) := \int_0^\infty \left(e^{-s\frac{x^2}{2}} - |\cos x|^s\right) \frac{dx}{x^{p+1}} \geqslant 0,$$

which again is an inequality of the kind $\int_{\mathcal{X}}(f^s - g^s)\,d\mu \geqslant 0$, only now with $\mathcal{X} = (0, +\infty)$, $f(x) = e^{-x^2/2}$, $g(x) = |\cos x|$, and $d\mu = \dfrac{dx}{x^{p+1}}$.

Unfortunately, there is no hope to prove the inequality $H(p,s) \geqslant 0$ for every $p \in [p_0, 2)$, $s \geqslant 1$ by our distribution function technique. Indeed, since $H(p_0, 2) = 0$, such a proof of the estimate $H(p_0, s) > 0$ for $s > 2$ would automatically be a proof of the estimate $H(p_0, s) < 0$ for $s < 2$ and vice versa. Fortunately, it is still possible to show that $H(p,s) \geqslant 0$ for $s \geqslant 2$, which is enough to establish Haagerup's theorem for the case when all the coefficients a_k do not exceed $\frac{1}{\sqrt{2}}$.

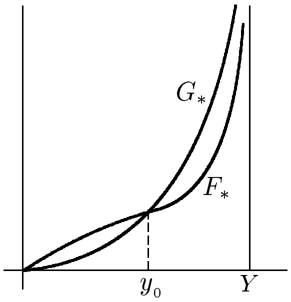
Step 4. Modified lemma on distribution functions and the proof of Haagerup's integral inequality.

Our lemma on distribution functions from Chapter I cannot be applied directly now because the corresponding distribution functions $F(y)$ and $G(y)$ are infinite for small values of y. Nevertheless, the *modified distribution functions*

$$F_*(y) = \mu\{x \in \mathcal{X} : f(x) < y\} \quad \text{and} \quad G_*(y) = \mu\{x \in \mathcal{X} : g(x) < y\}$$

are finite for all $0 < y < 1$, so we may use the following

Modified lemma on distribution functions. *Let $Y > 0$. Let $f, g : \mathcal{X} \to [0, Y]$ be any two measurable functions on a measure space $(\mathcal{X}, \mu)$. Let F_* and G_* be their modified distribution functions. Assume that both $F_*(y)$ and $G_*(y)$ are finite for every $y \in (0, Y)$. Assume also that at some point y_0 the difference $F_* - G_*$ changes sign from $+$ to $-$, i.e., $F_*(y) \geqslant G_*(y)$ for all $y \in (0, y_0)$ and $F_*(y) \leqslant G_*(y)$ for all $y \in (y_0, Y)$.*



Let again $S = \{s > 0 : f^s - g^s \in L^1(\mathcal{X}, \mu)\}$. Then the function

$$\varphi(s) = \frac{1}{s y_0^s} \int_{\mathcal{X}} (f^s - g^s)\, d\mu$$

is increasing on S. In particular, if $\int_{\mathcal{X}}(f^{s_0} - g^{s_0})d\mu \geqslant 0$, then $\int_{\mathcal{X}}(f^s - g^s)d\mu \geqslant 0$ for each $s > s_0$.

The proof is essentially the same as in Chapter I.

Taking $\mathcal{X}$, μ, f, g as above and putting $Y=1$, $s_0=2$, we see that to prove Haagerup's integral inequality in the desired range $p_0 \leqslant p \leqslant 2$, $s \geqslant 2$, we need only show that the corresponding modified distribution functions F_* and G_* satisfy the conditions of the modified lemma. These functions are not hard to compute: for $y \in (0,1)$, we have

$$F_*(y)=\mu\{x>0\,:\,e^{-\frac{x^2}{2}}<y\}=\mu\left(\sqrt{2\ln\tfrac{1}{y}},+\infty\right)=\int_{\sqrt{2\ln(1/y)}}^{+\infty}\frac{dx}{x^{p+1}}=\frac{1}{p}(2\ln\tfrac{1}{y})^{-p/2},$$

$$G_*(y)=\mu\{x>0\,:\,|\cos x|<y\}=\mu\bigcup_{k=0}^{\infty}(\pi k+\arccos y,\pi k+\pi-\arccos y)$$
$$=\sum_{k=0}^{\infty}\int_{\pi k+\arccos y}^{\pi k+\pi-\arccos y}\frac{dx}{x^{p+1}}=\frac{1}{p}\sum_{k=0}^{\infty}\left(\frac{1}{(\pi k+\arccos y)^p}-\frac{1}{(\pi k+\pi-\arccos y)^p}\right).$$

Make (decreasing) change of variable $y=\cos t$, $t\in(0,\frac{\pi}{2})$. Then our aim is to show that the difference $L(t)^{-p/2}-S(t)$, where

$$L(t)=(pF_*(\cos t))^{-2/p}=2\ln\frac{1}{\cos t},$$
$$S(t)=pG_*(\cos t)=\frac{1}{t^p}-\frac{1}{(\pi-t)^p}+\frac{1}{(\pi+t)^p}-\frac{1}{(2\pi-t)^p}+\frac{1}{(2\pi+t)^p}-\dots,$$

changes its sign only once on the interval $(0,\frac{\pi}{2})$, this change being from $-$ to $+$, or, which is the same, that the difference

$$\Phi(t)=L(t)-S(t)^{-p/2}$$

changes its sign only once on the interval $(0,\frac{\pi}{2})$, this change being from $+$ to $-$.

The behavior of $\Phi(t)$ on the interval $(0,\frac{\pi}{2})$ is quite complicated: there is no obvious monotonicity or concavity that would allow us to give a simple proof of the desired property by *real variable methods.* Nevertheless our functions are simpler than the distribution functions from the proof of Ball's integral inequality in another respect: they both can be extended analytically to the complex plane. So, it seems reasonable to try complex variable methods. We will prove the following

Lemma on zeroes. *Let $1<p<2$. Then the analytic extension $\Phi(z)$ of the difference $\Phi(t)$ to the vertical strip $\{0<\operatorname{Re} z<\frac{\pi}{2}\}$ has exactly one zero in that strip. This zero z_0 is real and the sign of $\Phi(t)$ changes from $+$ to $-$ at z_0.*

Proof. Since the minuend $L(t)$ and the subtrahend $S(t)^{-2/p}$ involve logarithms and fractional powers, some comment is due on how the corresponding branches in their extensions $L(z)$ and $S(z)^{-2/p}$ must be chosen. Since our complex-valued functions $L(z)$ and $S(z)$ of complex variable z are going to be extensions of the real-valued functions $L(t)$ and $S(t)$ of real variable t, we should have the symmetry $L(\overline{z})=\overline{L(z)}$ and $S(\overline{z})=\overline{S(z)}$. So, it is enough to trace the branches in the upper half-strip, i.e., when $\operatorname{Im} z\geqslant 0$.

For every such z, we have $\operatorname{Im}\cos z = (e^{-\operatorname{Im} z} - e^{\operatorname{Im} z})\sin\operatorname{Re} z < 0$. So, $\cos z$ stays in the lower half-plane and, thereby, $\frac{1}{\cos z}$ stays in the upper half-plane as z runs over the upper half-strip. Besides, the values of $L(z)$ on the interval $(0, \frac{\pi}{2})$ should be real. Thus, the branch of logarithm we need to choose is the one with the imaginary part between 0 and π.

Let us now turn to $S(z)$. Note first of all that for $p > 1$, the series in the definition of $S(z)$ converges absolutely for every z in the upper half-strip. Now observe that for every z in the upper half-strip, all the complex numbers z, $\pi + z$, $2\pi + z, \ldots$ belong to the right upper quadrant $\{\operatorname{Re}\zeta > 0, \operatorname{Im}\zeta \geqslant 0\}$. The branch of the function $\zeta \to \zeta^{-p}$ that is positive on the positive real semiaxis sends this quadrant to the angle $\Delta_p := \{-\frac{\pi p}{2} < \arg\zeta \leqslant 0\}$ (since $p < 2$, this angle is a subset of the lower half-plane). On the other hand, the numbers $\pi - z$, $2\pi - z, \ldots$ belong to the right lower quadrant, which is sent by the mapping $\zeta \to -\zeta^{-p}$ to the same angle Δ_p. So, the whole sum $S(z)$ stays in this angle when z runs over the upper half-strip and the branch of the power $-2/p$ in the subtrahend $S(z)^{-2/p}$ should be the one that sends the angle Δ_p to the upper half-plane (which is the "natural" branch in this angle).

Note now that the functions $L(z)$ and $S(z)$ are analytic everywhere in the closed strip $\{0 \leqslant \operatorname{Re} z \leqslant \frac{\pi}{2}\}$ except two singular points 0 and $\frac{\pi}{2}$. Let us look at what happens when z approaches these singularities.

When $z \to 0$, we have $L(z) = 2\ln\frac{1}{\cos z} = z^2 + \frac{1}{6}z^4 + o(z^4)$ while $S(z) = z^{-p} + O(z) = z^{-p}(1 + o(z^2))$ because $p > 1$. Thus

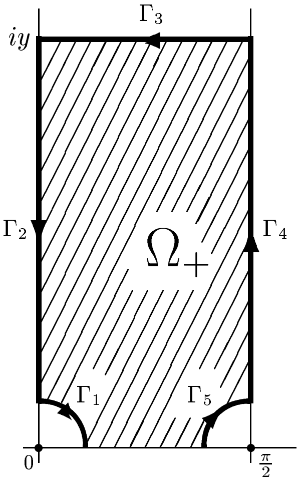
$$\Phi(z) = z^2 + \frac{z^4}{6} + o(z^4) - z^2(1 + o(z^2)) = \frac{z^4}{6} + o(z^4) \qquad \text{as } z \to 0.$$

Note now that $\frac{1}{\cos z} = \frac{1}{\sin(\frac{\pi}{2} - z)} = O\Big(\frac{1}{|\frac{\pi}{2} - z|}\Big)$ and $L(z) = O\Big(\ln\frac{1}{|\frac{\pi}{2} - z|}\Big)$ as $z \to \frac{\pi}{2}$. On the other hand, $\frac{\pi}{2}$ is a simple zero of the function $S(z)$ and therefore $S(z) = D\,(\frac{\pi}{2} - z)(1 + o(1))$ where $D = -S'(\frac{\pi}{2}) = 2p\left[\left(\frac{2}{\pi}\right)^{p+1} + \left(\frac{2}{3\pi}\right)^{p+1} + \ldots\right] > 0$. Thus

$$\Phi(z) \approx -D^{-2/p}\left(\frac{\pi}{2} - z\right)^{-2/p} \qquad \text{as } z \to \frac{\pi}{2}.$$

From these asymptotic relations, we can conclude that the function $\Phi(t)$ is positive near 0 and negative near $\frac{\pi}{2}$. So it has at least one root on the interval $(0, \frac{\pi}{2})$ and, if this root is unique, changes its sign from + to − there.

Now let us estimate the increment of the argument of the function $\Phi(z)$ over the boundary of the region Ω, the upper half Ω_+ of which is shown on the picture below.



Note that $\Delta_{\partial\Omega} \arg \Phi = 2\Delta_{\partial\Omega_+} \arg \Phi$ due to the symmetry $\Phi(\overline{z}) = \overline{\Phi(z)}$. The boundary $\partial\Omega_+$ consists of 5 parts $\Gamma_1, \ldots, \Gamma_5$. We assume that the radii of the circle arcs Γ_1 and Γ_5 are very small and the height y of Γ_3 is very large.

Estimates for Γ_1 and Γ_5. From the asymptotic relations above, we immediately obtain that the function $\Phi(z)$ does not vanish on Γ_1 and Γ_5 and

$$\Delta_{\Gamma_1} \arg \Phi \approx -2\pi\,; \qquad \Delta_{\Gamma_5} \arg \Phi \approx \frac{\pi}{p}.$$

Estimate for Γ_3. Clearly,

$$L(z) = 2\ln\frac{1}{\cos z} = 2\ln\frac{2}{e^{iz}+e^{-iz}} = 2iz + O(1) \quad \text{as } \operatorname{Im} z \to +\infty,\ z \in \Omega_+.$$

As to $S(z)$, we have

$$\operatorname{Im}\frac{1}{z^p},\ \operatorname{Im}\frac{1}{(\pi+z)^p}, \cdots < 0 \quad \text{and} \quad \operatorname{Im}\frac{1}{(\pi-z)^p},\ \operatorname{Im}\frac{1}{(2\pi-z)^p}, \cdots > 0$$

for every $z \in \Omega_+$. What is more, the absolute values of the imaginary parts of the first $\operatorname{Im} z \approx |z|$ terms in the series for $S(z)$ are not less than $C|z|^{-p}$ with some absolute $C > 0$. Hence, $|S(z)| \geqslant |\operatorname{Im} S(z)| \geqslant \mathrm{Const}\,|z|^{1-p}$ and, thereby,

$$S(z)^{-2/p} = O(|z|^{2(p-1)/p}) = o(|z|)$$

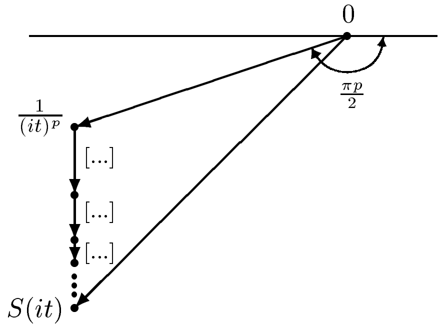
on Γ_3 (recall that $p < 2$). Thus $\Phi(z) \approx 2iz$ on Γ_3, so Φ does not vanish on Γ_3 and

$$\Delta_{\Gamma_3} \arg \Phi \approx 0.$$

Estimate for Γ_2. For every $z = it$, $t > 0$ on the vertical side Γ_2, we have $L(z) = 2\ln\frac{2}{e^{-t}+e^{t}} \in \mathbb{R}$. Representing the function $S(z)$ as

$$S(z) = S(it) = \frac{1}{(it)^p} - \left[\frac{1}{(\pi - it)^p} - \frac{1}{(\pi + it)^p}\right] - \left[\frac{1}{(2\pi - it)^p} - \frac{1}{(2\pi + it)^p}\right] - \dots$$

and observing that every difference in brackets is a "positive imaginary" number, we arrive at the following picture:



Hence $0 < \arg\left[S(it)^{-2/p}\right] < \pi$, i.e., $\operatorname{Im}\left[S(it)^{-2/p}\right] > 0$. Therefore $\operatorname{Im}\Phi(it) < 0$ for all $t > 0$, or, by other words, $\Phi(z)$ stays in the (open) lower half-plane as z runs over Γ_2. Therefore Φ does not vanish on Γ_2 and

$$\Delta_{\Gamma_2} \arg\Phi \leqslant \pi.$$

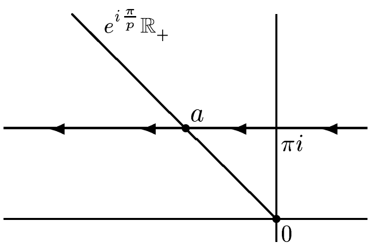
Estimate for Γ_4. This is the most interesting part of the contour $\partial\Omega_+$. On Γ_4 we have $z = \frac{\pi}{2} + it,\ t > 0$. Hence

$$L(z) = 2\ln\frac{2}{ie^{-t} - ie^{t}} = 2\ln i + 2\ln\frac{2}{e^t - e^{-t}} = \pi i + 2\ln\frac{2}{e^t - e^{-t}}.$$

Meanwhile,

$$S(z) = \left[\frac{1}{(\frac{\pi}{2} + it)^p} - \frac{1}{(\frac{\pi}{2} - it)^p}\right] + \left[\frac{1}{(\frac{3\pi}{2} + it)^p} - \frac{1}{(\frac{3\pi}{2} - it)^p}\right] + \dots.$$

Every difference in brackets is a "negative imaginary" number. Thus $S(z) \in -i\mathbb{R}_+$ and thereby $S(z)^{-2/p} \in e^{i\pi/p}\,\mathbb{R}_+$. So, when t runs up from 0 to $+\infty$, the minuend $L(\frac{\pi}{2} + it)$ runs over the horizontal line $\operatorname{Im} z = \pi$ from right to left, while the subtrahend $S(\frac{\pi}{2} + it)^{-2/p}$ wanders over the ray $\arg z = \frac{\pi}{p}$ (see the picture below).



As long as $L(\frac{\pi}{2}+it)$ remains to the right from this ray, the difference $\Phi = L - S^{-2/p}$ stays in the "upper right" half-plane bounded by the line $e^{i\pi/p}\,\mathbb{R}$. So, the corresponding increment of $\arg\Phi$ does not exceed π. Once $L(\frac{\pi}{2}+it)$ has crossed this ray, it never leaves the "left lower" half-plane bounded by the same line and again $\arg\Phi$ cannot change by more than π. Thus, if Φ does not vanish on Γ_4, we have

$$\Delta_{\Gamma_4}\Phi \leqslant 2\pi.$$

Bringing these estimates together and recalling that $\Delta_{\partial\Omega}\arg\Phi = 2\,\Delta_{\partial\Omega_+}\arg\Phi$, we see that in this case the total change of $\arg\Phi$ over the boundary $\partial\Omega$ does not exceed ("up to epsilon") $2(-2\pi+\pi+0+2\pi+\frac{\pi}{p}) = 2\pi(1+\frac{1}{p})$. For $p>1$, this quantity is strictly less than 4π and thereby there can be no second root of Φ in the strip $0<\operatorname{Re}z<\frac{\pi}{2}$.

Suppose now that Φ vanishes at some point $z_0\in\Gamma_4$. Replace a small interval of Γ_4 centered at z_0 by the clockwise oriented semicircle bypassing z_0 from the left. Then we shall get an extra *negative* change of argument by π (if the root is simple) or more (if it is multiple) when going over this semicircle. And the more roots we have and the higher their multiplicity is, the more we gain on the negative side! [2] So if Φ had any zeroes on Γ_4, we would end up with even better estimate $\Delta_{\partial\Omega}\arg\Phi < 2\pi$, which would prohibite Φ to have roots in the strip at all! But we know that at least one such root exists, ruling this possibility out.

Haagerup's integral inequality is thus completely proved. □

Step 5. What to do if one of the coefficients a_k is large?

From now on, it will be more convenient to replace our usual normalization condition $\sum_{k=1}^n a_k^2 = 1$ by the condition $a_1 = 1$. Our aim is to prove the inequality

$$R_p(a) = \int_0^1 \left|\sum_{k=1}^n a_k r_k(t)\right|^p dt \geqslant A_p(1+a_2^2+\cdots+a_n^2)^{p/2} =: A_p\,\varphi_p(a_2^2+\cdots+a_n^2)$$

[2] Actually this last remark is empty because from what we said above, it is clear that Φ can vanish on Γ_4 only at the point where $L=a$ and only if $S(\frac{\pi}{2}+it)^{-2/p}$ reaches the point a simultaneously with $L(\frac{\pi}{2}+it)$. Also, this root must be simple because the speed at which $L(\frac{\pi}{2}+it)$ runs over the line $\mathbb{R}+i\pi$ never vanishes.

with $A_p = 2^{\frac{p-2}{2}}\frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{3}{2})}$ for all $p \in [p_0, 2)$, $a = (a_1, \dots, a_n) \in \mathbb{R}^n$. (Here and below $\varphi_p(x)$ stands for $(1+x)^{p/2}$)

The idea of the proof We already know that the desired inequality holds when $\max_k a_k^2 \leqslant \frac{1}{2}\sum_k a_k^2$ (this is just the result of Step 4, only slightly reformulated). How to get the desired estimate in full generality? We will try to use the almost obvious formula

$$R_p(a) = \frac{R_p(a^+) + R_p(a^-)}{2}$$

where $a^\pm = (a_1, \dots, a_{n-2}, a_{n-1} \pm a_n)$ $(n \geqslant 3)$ to reduce the bad case when one of the coefficients is large to the case when the number of coefficients is $n-1$ instead of n. Formally that means that we are going to use induction on n. Since for $n \geqslant 3$ we have the identity

$$a_2^2 + \dots + a_n^2 = \frac{1}{2}\left\{\left[(a_2^-)^2 + \dots + (a_{n-1}^-)^2\right] + \left[(a_2^+)^2 + \dots + (a_{n-1}^+)^2\right]\right\},$$

the induction step would be fairly easy *if the function* $\varphi(x) = (1+x)^{p/2}$ *were convex.* Then, denoting for brevity $a_2^2 + \dots + a_n^2$ by x and $(a_2^\pm)^2 + \dots + (a_{n-1}^\pm)^2$ by $x^\pm$, we could make the induction step in just one line:

$$R_p(a) = \frac{R_p(a^+) + R_p(a^-)}{2} \overset{\substack{\text{induction}\\ \text{assumption}}}{\geqslant} A_p\,\frac{\varphi_p(x^+) + \varphi_p(x^-)}{2}$$
$$\overset{\text{convexity}}{\geqslant} A_p\,\varphi_p\left(\frac{x^+ + x^-}{2}\right) = A_p\varphi_p(x).$$

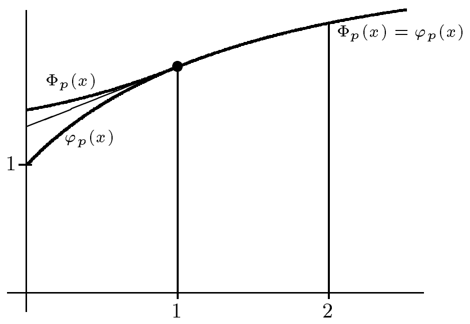
Unfortunately there is one small, but annoying obstacle on this otherwise excellent way: for $p < 2$ the function $\varphi_p(x)$ is *concave*! When proving inequalities by induction, the standard trick that allows to get rid of such “small but annoying” obstacles is to strengthen the statement. In our case it would be ideal to find a convex function $\Phi_p(x)$ that is pointwise not less than $\varphi_p(x)$ and to prove the desired estimate with Φ_p in place of φ_p, i.e., to prove the estimate

$$R_p(a) \geqslant A_p\,\Phi_p(a_2^2 + \dots + a_n^2). \qquad (**)$$

Unfortunately, there is no finite function $\Phi_p(x)$ dominating $\varphi_p(x)$ and convex on the whole ray $[0, +\infty)$ either. But we do not actually need that much: we can always reshuffle the coefficients $a_1, \dots, a_n$ in the original inequality so that $a_1 = 1$ is the largest of them and then just to refer to the result of Step 4 when there is no large coefficient, i.e., when $x \geqslant 1$. So, it seems that we only need to change our function $\varphi_p(x)$ on the interval $[0, 1]$. It is really the case and the function that will work is

$$\Phi_p(x) = \begin{cases} \varphi_p(x), & \text{if } x \geqslant 1; \\ 2\varphi_p(1) - \varphi_p(2-x), & \text{if } 0 \leqslant x \leqslant 1. \end{cases}$$

In other words, Φ_p is the function, whose graph is obtained by replacing the part of the graph of φ_p on $[0,1]$ by the symmetric image of the graph of φ_p on $[1,2]$ with respect to the point $(1,\varphi_p(1)) = (1,2^{p/2})$ (see the picture below).



On the interval $[0,1)$, the graphs of the functions Φ_p and φ_p are separated by their common tangent line at the point $(1,\varphi_p(1))$ and therefore $\Phi_p(x) > \varphi_p(x)$ when $0 \leqslant x < 1$, and $\Phi_p(x) = \varphi_p(x)$ when $x \geqslant 1$. Now we are ready to make

THE INDUCTION STEP. Observe first of all that

$$\Phi_p\left(\frac{x'+x''}{2}\right) \leqslant \frac{\Phi_p(x')+\Phi_p(x'')}{2} \quad \text{whenever} \quad x',x'' \geqslant 0 \text{ and } \frac{x'+x''}{2} \leqslant 1.$$

(This property is a little bit stronger than just convexity on $[0,1]$, because we do not assume that $\max\{x',x''\} \leqslant 1$. It was the desire to have this "extended convexity" that prevented us from just replacing the part of the graph of φ_p on $[0,1]$ by the corresponding piece of the tangent line).

Indeed, we may assume without loss of generality that $x' < x''$. If $x'' \leqslant 1$, the desired inequality follows immediately from the convexity of Φ_p on $[0,1]$. Suppose now that $x'' > 1$. Notice that

$$\frac{\partial}{\partial x'}\Phi_p\left(\frac{x'+x''}{2}\right) = \frac{1}{2}\left.\frac{d\Phi_p}{dx}\right|_{x=\frac{x'+x''}{2}} \geqslant \frac{1}{2}\left.\frac{d\Phi_p}{dx}\right|_{x=x'} = \frac{\partial}{\partial x'}\frac{\Phi_p(x')+\Phi_p(x'')}{2}$$

whenever $x' < x''$, $\frac{x'+x''}{2} \leqslant 1$, because the derivative Φ_p' is increasing on the interval $[0,1]$. Therefore the function $\Phi_p(\frac{x'+x''}{2})$ increases in x' faster then $\frac{\Phi_p(x')+\Phi_p(x'')}{2}$, and it is enough to check our inequality for the maximal possible value of x', i.e., for $x' = 2 - x''$. But in this case our inequality turns into identity due to the definition of Φ_p.

Now suppose that $n \geqslant 3$ and that the estimate $(**)$ is already established for linear combinations of $n-1$ Rademacher functions. Let $a = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$, $a_1 = 1$. Denote as before $x = a_2^2 + \dots + a_n^2$. Three cases are possible:

- $a_1 = 1$ is not the largest coefficient. Then $x > 1$ and $(**)$ coincides with the homogeneous estimate $R_p(a) \geqslant A_p(a_1^2 + \cdots + a_n^2)^{p/2}$. Rearranging the coefficients in the order of decreasing absolute value (this doesn't change either part of the homogeneous estimate) and renormalizing so that the first coefficient is again 1, we shall arrive at some new version of the inequality $(**)$ in which $a_1 = 1$ is the largest coefficient. This new version is not weaker than the homogeneous estimate, i.e., not weaker than the original version we wanted to prove.

- $a_1 = 1$ is the largest coefficient, $x \geqslant 1$. In this case, $(**)$ still coincides with the homogeneous estimate and $\max_k a_k^2 = 1 \leqslant \frac{1}{2}(1+x) = \frac{1}{2}\sum_k a_k^2$. So we may just refer to Step 4.
- $a_1 = 1$ is the largest coefficient, $x < 1$. Then (in the same notation as above)

$$R_p(a) = \frac{R_p(a^+) + R_p(a^-)}{2} \overset{\substack{\text{induction}\\ \text{assumption}}}{\geqslant} A_p\,\frac{\Phi_p(x^+) + \Phi_p(x^-)}{2}$$
$$\geqslant A_p\,\Phi_p\left(\frac{x^+ + x^-}{2}\right) = A_p\Phi_p(x)$$

(because $x^{\pm} \geqslant 0$ and $\frac{x^- + x^+}{2} = x < 1$).

So it remains only to prove

THE INDUCTION BASE $n = 2$. We need to prove the inequality

$$R_p(1, a_2) \geqslant A_p\Phi_p(a_2^2).$$

Again we may assume that $a_1 = 1$ is the largest of the two coefficients, i.e., $a_2^2 = x \leqslant 1$ (otherwise, after exchanging a_1 and a_2 and renormalizing, we shall obtain a stronger inequality to prove). Computing the left hand part and recalling the definition of $\Phi_p(x)$ and our hypothetical formula for A_p, we can rewrite it as

$$\frac{(1+\sqrt{x})^p + (1-\sqrt{x})^p}{2} \geqslant 2^{\frac{p-2}{2}}\,\frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{3}{2})}\left[2\cdot 2^{p/2} - (3-x)^{p/2}\right] \qquad (0 \leqslant x \leqslant 1).$$

The ratio $\frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{3}{2})}$ does not exceed 1 for all $p \in [p_0, 2)$, so it is enough to prove the estimate

$$\frac{(1+\sqrt{x})^p + (1-\sqrt{x})^p}{2} \geqslant 2^{\frac{p-2}{2}}\left[2\cdot 2^{p/2} - (3-x)^{p/2}\right] = 2^{p-1}\left[2 - \left(\frac{3-x}{2}\right)^{p/2}\right]$$

At first glance, this strengthening of the statement to prove may seem completely unmotivated, but if we recall that for $p = p_0$ this ratio equals 1 and that we do not know (or, at least, do not want to know) what p_0 is, it becomes quite natural.

Dividing both parts by 2^{p-1} and transferring $\left(\frac{3-x}{2}\right)^{p/2}$ to the left hand side, we can restate our estimate as

$$\left(\frac{1+\sqrt{x}}{2}\right)^p + \left(\frac{1-\sqrt{x}}{2}\right)^p + \left(\frac{3-x}{2}\right)^{p/2} \leqslant 2.$$

Now denote

$$a = \frac{1+\sqrt{x}}{2}, \qquad b = \frac{1-\sqrt{x}}{2}.$$

Taking into account that $1-x = 4ab$, we can rewrite our inequality as

$$a^p + b^p + (1+2ab)^{p/2} \leqslant 2 \qquad (p \leqslant 2;\ a, b \geqslant 0;\ a+b = 1).$$

Since for $p = 2$ we have an identity and since the left hand part is *convex* in p, it is sufficient to show that

$$\left.\frac{d}{dp}\right|_{p=2} \left[a^p + b^p + (1+2ab)^{p/2}\right] \leqslant 0,$$

or, which is the same, that

$$a^2 \ln a + b^2 \ln b + \frac{1}{2}(1+2ab)\ln(1+2ab) \leqslant 0.$$

Note now that $\frac{d^2}{dt^2}[(1+t)\ln(1+t)] = \frac{1}{1+t} \leqslant 1$ for every $t \geqslant 0$. Thereby, using the second order Taylor formula at the point $t_0 = 0$, we obtain $(1+t)\ln(1+t) \leqslant t + \frac{1}{2}t^2$. In particular,

$$\frac{1}{2}(1+2ab)\ln(1+2ab) \leqslant \frac{1}{2}\left(2ab + \frac{1}{2}(2ab)^2\right) = ab + (ab)^2.$$

On the other hand, $a^2 \ln a = a^2 \ln(1-b) = -a^2\left(b + \frac{b^2}{2} + \dots\right) \leqslant -a^2\left(b + \frac{b^2}{2}\right)$ and, similarly, $b^2 \ln b \leqslant -b^2\left(a + \frac{a^2}{2}\right)$. Thus

$$a^2 \ln a + b^2 \ln b \leqslant -a^2\left(b + \frac{b^2}{2}\right) - b^2\left(a + \frac{a^2}{2}\right) = -ab(a+b) - (ab)^2 = -ab - (ab)^2.$$

Haagerup's theorem (case $p < 2$) is thus completely proved. $\square$

References

[1] K. Ball, *Cube slicing in* $\mathbb{R}^n$, Proc. Amer. Math. Soc., **97 (3)** (1986), 465–473.

[2] K. Ball, *Some remarks on the geometry of convex sets,* Geometric aspects of functional analysis (1986/87), 251–260, Lecture Notes in Math., **1317**, Springer, Berlin-New York (1988).

[3] H. Busemann, C. M. Petty, *Problems on convex bodies,* Math. Scand., **4** (1956), 88–94.

[4] R. J. Gardiner, *A positive answer to the Busemann-Petty problem in three dimensions,* Ann. of Math. (2) **140** (1994), no. 2, 435–447.

[5] U. Haagerup, *The best constants in the Khintchine inequality,* Studia Math., **70 (3)** (1982), 231–283.

[6] G. Y. Zhang, *Intersection bodies and the Busemann-Petty inequalities in* $\mathbb{R}^4$, Ann. of Math. (2) **140** (1994), no. 2, 331–346.

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