On the behavior of a trigonometric polynomial with given number of terms on small subsets of the unit circumference ("norm duplication" problem)

§0. Introduction

This paper can be considered as an appendix to my previous one* , where I have obtained some rather sharp estimates for the norm of an exponential polynomial on the interval $I \subset \mathbb{R}$ through its number of terms and its norm on some small measurable subset $E \subset I$. After publishing several people asked me whether I can obtain some good estimates in the case when the measure of E is not small, but, inversely, very close to the measure of I. Usually I answered "I don't know" and forgot of it. But at last the question has become interesting for myself. It turns out that it is rather difficult and I cannot answer it in general case still. Nevertheless a little modification of the technique used in [1] allows to solve one particular problem which seems to be of special interest for applications : the so-called "norm duplication" problem** .For simplicity I will deal with the periodic case below, though almost all results remain valid in the more general case of "well-separated" exponents*** (The only exception is the case of L^∞-norm where we essentially use the fact that the circumference has neither begin nor end point. But even in this case it is rather due to my poor fantasy in inventing polynomial solutions of corresponding interpolation problem than to bad choice of the method of proof itself). If we impose no separation condition on the exponents, the boundary effect begins to play a crucial role and the problem becomes much more complicated. So, I'm not going to discuss it here. (Actually I can prove some partial results, but in general the problem remains open). At last I'd like to note that I have tried to make the article as self-contained and easy to read as possible. So, I will often give rather rough but simple estimates and examples, just mentioning the best those which I can obtain.

§1. Formulation of the problem.

First of all let's describe the objects we are going to deal with. A trigonometric

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* "Local estimates of exponential polynomials and their applications to inequalities of the Uncertainty Principle type", published in Algebra and Analysis, v.5, No 4, 1993.

** Roughly speaking, the question is how close must the measure of E be to that of I to guarantee that the norm of the polynomial on the whole interval I not more than twice exceeds its norm on E. Another way to formulate this question is how small can be "essential support" of a polynomial of order n.

*** Really I need only the Riesz basis property : $\left\|\sum c_j e^{2\pi i\lambda_j t}\right\|_{L^2(I)} \asymp \left(\sum |c_j|^2\right)^{1/2}$.

polynomial of order n is a finite sum $p(z) = \sum_{k=1}^{n} c_k z^{m_k}$ where $c_k \in \mathbb{C},\ m_1 < \ldots < m_n \in \mathbb{Z}$ (it is considered as a function from the unit circumference $\mathbb{T} = \{z : |z| = 1\}$ into $\mathbb{C}$). We denote by μ the Lebesgue measure on $\mathbb{T}$, normalized by $\mu(\mathbb{T}) = 1$. For any (measurable) subset $H \subset \mathbb{T}$ of positive measure and for any (continuous, say) function $f : \mathbb{T} \to \mathbb{C}$ we can define the L^q-mean of $|f|$ over H by $\|f\|_{L^q(H)} = \left\{ \frac{1}{\mu(H)} \int_H |f|^q d\mu \right\}^{1/q}$ $(0 < q < \infty)$. If we take the limits of this expression when q tends to 0 and to $+\infty$, we shall get correspondently $\|f\|_{L^0(H)} = \exp\left\{ \frac{1}{\mu(H)} \int_H \log|f| d\mu \right\}$ (the geometric mean of $|f|$) and $\|f\|_{L^\infty(H)} = \sup_H |f|$ (this notation is quite usual). Fix some $q \in [0, +\infty]$. The question we will answer is what conditions are to be imposed on n and ν to guarantee that

$$\|p\|_{L^q(\mathbb{T})} \le B\, \|p\|_{L^q(\mathbb{T}\setminus E)} \qquad (*)$$

for each trigonometric polynomial p of order n and for each set $E \subset \mathbb{T}$ satisfying $\mu(E) \le \nu$ (B is some positive constant). By other words, if we call a pair (n, ν) satisfying $(*)$ B-good, the problem is to describe all sets of pairs $\Omega = \{(n, \nu)\} \subset \mathbb{N} \times (0, 1)$ such that there exists some constant $B > 0$ for which all pairs in Ω are B-good (we shall call them good sets). It follows immediately from the definition that if (n, ν) is a B-good pair then every pair (n', ν') with $n' \le n,\ \nu' \le \nu$ is B-good as well. So we may assume that every good set is a set of the kind $\Omega = \{(n, \nu) : n \in \mathbb{N},\ \nu < \nu_n\}$ where ν_n is some nonincreasing sequence of positive numbers (at any rate each good set is a subset of a good set of this kind). As the absolute value of the trigonometric polynomial of order 1 is identically equal to some constant on $\mathbb{T}$, there is no restriction to be imposed on ν_1 (we can take $\nu_1 = 1$). If we consider the polynomial $p(z) = 1 - z$ of order 2 we shall easily find that ν_2 is to be strictly less than 1. Now we are able to formulate the "main proposition" of the article.

Theorem :

Let ν_n be some nonincreasing sequence of positive numbers such that $\nu_2 < \nu_1 = 1$. The set of pairs $\Omega = \{(n, \nu) : n \in \mathbb{N},\ \nu < \nu_n\}$ is good if and only if

$$\begin{cases} \nu_n = O(\frac{1}{n}), & \text{when } 0 \le q \le 2 \\ \nu_n = O(\frac{1}{n^{q/2}}), & \text{when } 2 \le q < +\infty \\ \nu_n = O(e^{-\delta n}) \text{ with some } \delta > 0, & \text{when } q = +\infty\ . \end{cases} \qquad (*)$$

§2. Necessity.

It will be convenient to consider 4 cases separately : $q = 0$, $0 < q \le 2$, $2 < q < +\infty$ and $q = +\infty$.

§ 2.1. q = 0.

Consider a polynomial $p(z) = (1+z)^{n-1}$ and the set $E = \{|z-1| \le \nu\}$ $(0 < \nu < 1/2)$ of measure $\mu(E) = \frac{2\arcsin \nu/2}{\pi} \in \left(\frac{\nu}{\pi}, \frac{\nu}{2}\right)$. As $p(z)$ is an analytic function having no zeros inside the unit disk $\mathbb{D}$, we have $\|p\|_{L^0(\mathbb{T})} = |p(0)| = 1$.

From other side, $\|p\|_{L^0(\mathbb{T}\setminus E)} = \exp\left\{\frac{1}{\mu(\mathbb{T}\setminus E)} \int\limits_{\mathbb{T}\setminus E} \log|p|d\mu\right\} =$

$= \exp\left\{-\frac{1}{\mu(\mathbb{T}\setminus E)} \int\limits_{E} \log|p|d\mu\right\} \le \exp\{-(n-1)\mu(E)\log(2-\nu)\} \le e^{-\frac{\nu n}{100}}$, say (provided that $n > 1$, of course). If the pair (n,ν) is B-good, we get $1 \le B\,e^{-\frac{\nu n}{100}}$ and consequently $\nu \le \frac{100\log B}{n}$.

§2.2. 0 < q ≤ 2.

Consider the polynomial $p(z) = \left(\frac{z^m - 1}{z-1}\right)^r$ where $m = \left[\frac{nq}{4}\right]$, $r = \left[\frac{4}{q}\right]$ ($[x]$ denotes the entire part of x). It is well-defined for $n > 4/q$ and we have

$$\|p\|^q_{L^q(\mathbb{T})} = \int\limits_{\mathbb{T}} \left|\frac{z^m - 1}{z-1}\right|^{rq} = \int\limits_0^1 \left|\frac{e^{\pi imt} - 1}{e^{\pi it} - 1}\right|^{rq} dt \ge$$

$$\ge \int\limits_0^{1/m} \left|\frac{e^{\pi imt} - 1}{e^{\pi it} - 1}\right|^{rq} dt \ge \int\limits_0^{1/m} \left(\frac{2mt}{\pi t}\right)^{rq} = \frac{1}{m}\left(\frac{2m}{\pi}\right)^{rq}.$$

Let now E be the arc of measure $\nu < 1/2$ with the midpoint at 1. Then

$$\|p\|^q_{L^q(\mathbb{T}\setminus E)} = \frac{1}{1-\nu} \int\limits_{\nu}^{1} \left|\frac{e^{\pi imt}(-1)}{e^{\pi it} - 1}\right|^{rq} dt \le$$

$$\le 2\int\limits_{\nu}^{\infty} \left(\frac{2}{2t}\right)^{rq} dt = \frac{2}{rq-1}\,\nu^{-rq+1}$$

(as $q \le 2$ we have obviously $rq \in (2,4)$).

If (n,ν) is B-good we should have

$$\frac{1}{m}\left(\frac{2m}{\pi}\right)^{rq} \le B\,\frac{2}{rq-1}\,\nu^{-rq+1}$$

and therefore $\nu \le \frac{\text{Const}}{m} = O\left(\frac{1}{n}\right)$ again. Proceeding in a little more accurate (but much more boring) way, it is possible to prove that there exists some (small) absolute constant $a > 0$ such that for every triple $n \in \mathbb{N},\ \nu \in (0,1/2),\ q \in [0,2]$ one can find a polynomial $p(z)$ (of the kind $\left(\frac{z^m-1}{z-1}\right)^r$ with suitably chosen m and r satisfying $mr \le n$) and the arc $E = \left\{e^{2\pi it} : |t| \le \frac{\nu'}{2}\right\}$ of measure $\mu(E) = \nu' \le \nu$ such that

$$\|p\|_{L^q(\mathbb{T})} \ge e^{a(n-1)\nu}\,\|p\|_{L^q(\mathbb{T}\setminus E)}.$$

I firmly believe that, inversely, there exists some (large) absolute positive constant A such that the inequality

$$\|p\|_{L^q(\mathbb{T})} \le e^{A(n-1)\mu(E)}\,\|p\|_{L^q(\mathbb{T}\setminus E)} \quad (0 \le q \le 2)$$

holds for every polynomial of order n and every measurable subset $E \subset \mathbb{T}$ of measure $0 < \mu(E) \le 1/2$ but the case $q = 0$ is now the only one when I can prove it. It is very easy to prove the estimate desired for polynomials of degree n (or for arcs E) but I don't know what to do with the general case nevertheless !

§ **2.3.** $\mathbf{2 < q \le +\infty}$.

In this case I don't know how to construct an explicit example. We shall deal with random polynomials which have the desired behavior with positive probability instead.

Let's start with some auxiliary lemmas (all of them are well-known, so the reader familiar with the topic may just look at their formulations). Let integers n and N satisfy $1 < n < N < e^n$. Let $\xi_j\ \ (j = 0,\dots,N-1)$ be independent random variables defined by $P\{\xi_j = 1\} = \frac{n}{N} = \delta$, $P\{\xi_j = 0\} = 1-\delta$. (We use here the common notations $P\{A\}$ for the probability of an event A and $E\xi$ for the mathematical expectation of a random variable ξ).

Put $p(z) = p(z|\xi) = \sum\limits_{j=0}^{N-1} \xi_j z^j$.

Lemma 1 :

ord $p \le 2n$ with probability very close to 1 when n is large.

Proof.

Let $t \in (0,1)$. We have $\operatorname{ord} p = \sum_{j=0}^{N-1} \xi_j$, $E\, e^{t \operatorname{ord} p} = E\, e^{t \sum \xi_j} = \prod_{j=0}^{N-1} E\, e^{t\xi_j} = \{1 + \delta(e^t - 1)\}^N \le e^{\delta(e^t-1)N} = e^{n(e^t-1)}$.

So, $P\{\operatorname{ord} p \ge 2n\} \le e^{-2nt} E\, e^{t \operatorname{ord} p} \le e^{(e^t-1-2t)n}$. As $e^t - 1 - 2t < 0$ for small positive t, we obtain that this probability is exponentially small in n, proving the statement.

Let's decompose ξ_j into the sum $\delta + \eta_j$ where η_j are independent random variables, $P\{\eta_j = 1-\delta\} = \delta$, $P\{\eta_j = -\delta\} = 1 - \delta$ and therefore $E\eta_j = 0$.

$$p(z) = \delta \sum_{j=0}^{N-1} z^j + \sum_{j=0}^{N-1} \eta_j z^j = \delta\, \frac{z^N - 1}{z - 1} + Q(z).$$

The first term is just a Dirichlet kernel and its behavior is known to everybody. So let us calculate $L^q(\mathbb{T})$-norm of the second term $Q(z)$.

Lemma 2. (Khinchin inequality)

$$\|Q\|_{L^q(\mathbb{T})} \le \begin{cases} C_q\sqrt{n}, & 2 \le q < +\infty \\ C_\infty \sqrt{n \log N}, & q = +\infty \end{cases}$$

with probability 1/2, say.

Proof.

Let's estimate $\operatorname{Re} Q$ and $\operatorname{Im} Q$ separately.

For each $t \in (0,1)$ we have

$$E\, e^{t \operatorname{Re} Q(z)} = \prod_{j=0}^{N-1} E\, e^{t \operatorname{Re} z^j} \le \prod_{j=0}^{N-1} E\left(1 + t\eta_j \operatorname{Re} z^j + 2t^2\eta_j^2 (\operatorname{Re} z^j)^2\right)$$

(because $|t\eta_j \operatorname{Re} z^j| \le 1$ and $e^x \le 1 + x + 2x^2$ for every $x \in (-1,1)$)

$$\le \prod_{j=0}^{N-1} (1 + 2t^2 E\eta_j^2) = \left[1 + 2t^2\delta(1-\delta)\right]^N \le e^{2t^2\delta N} = e^{2nt^2}.$$

The same is true for $E\, e^{-t\,\mathrm{Re}\, Q(z)}$ and therefore

$$E\, e^{t|\,\mathrm{Re}\, Q|} \le E\, e^{t\,\mathrm{Re}\, Q} + E\, e^{-t\,\mathrm{Re}\, Q} \le 2e^{2nt^2}.$$

Note that $e^{x+y} \le e^{2x} + e^{2y}$ $(x, y > 0)$ and therefore

$$E\, e^{t|Q|} \le E\, e^{2t|\,\mathrm{Re}\, Q|} + E\, e^{2t|\,\mathrm{Im}\, Q|} \le 4e^{8nt^2} \ \text{ for } \ t \in (0, 1/2).$$

So $E \int\limits_{\mathbb{T}} e^{t|Q|} d\mu \le 4e^{8nt^2}$ and for each fixed $t \in (0, 1/2)$ $\int\limits_{\mathbb{T}} e^{t|Q|} d\mu \le 8e^{8nt^2}$ with probability $1/2$ or more. As $e^{tx} \ge \frac{t^q x^q}{q^q}$ for $t, x > 0$ and $q > 2$, we have $\int\limits_{\mathbb{T}} |Q|^q d\mu \le \frac{q^q}{t^q}\, 8e^{8nt^2}$ and, putting $t = \frac{1}{\sqrt{8n}}$, $\|Q\|_{L^q(\mathbb{T})} \le (8e)^{1/q} q\sqrt{8n}$ as has been promised. To estimate L^∞-norm just note that for any trigonometric polynomial Q of degree N the set of points at which it is at least a half as large as its maximum, has measure greater than $\frac{1}{2\pi N^2}$.

(Indeed, if $Q = \sum\limits_{j=0}^{N-1} c_j z^j$, say, and $M = \sup |Q| = |Q(z_0)|$ then $|c_k| \le M$ for every k and for every $z \in \mathbb{T}$

$$|Q(z)| \ge |Q(z_0)| - \sum_{j=0}^{N-1} |c_j|\, |z^j - z_0^j| \ge M - M \sum_{j=0}^{N-1} j|z - z_0| \ge M - MN^2|z - z_0| \ge \frac{M}{2}$$

when $|z - z_0| \le \frac{1}{2N^2}$).

So, $\frac{1}{2\pi N^2} \exp\left(\frac{t}{2}\, \|Q\|_{L^\infty(\mathbb{T})}\right) \le \int\limits_{\mathbb{T}} e^{t|Q|} d\mu \le 8e^{8nt^2}$ and thereby $\|Q\|_{L^\infty(\mathbb{T})} \le \frac{2}{t}\log(16\pi N^2) + 16\, tn$. Putting here $t = \frac{1}{2}\sqrt{\frac{\log N}{n}}$, say, we get the estimate desired (as N was supposed to be less than e^n, t is actually less than $1/2$).

Now let $2 < q < +\infty$.

Put $N = [(\varepsilon n)^{q/2}]$ where ε is some very small positive number to be chosen later.

So we can choose a polynomial $P(z)$ such that $\mathrm{ord}\, P \le 2n$ and

$$\|P\|_{L^q(\mathbb{T})} \ge \left\| \delta\, \frac{z^N - 1}{z - 1} \right\|_{L^q(\mathbb{T})} - \|Q\|_{L^q(\mathbb{T})} =$$

$$= \delta \left\{ \int_0^1 \left| \frac{e^{\pi i N t} - 1}{e^{\pi i t} - 1} \right|^q \right\}^{1/q} - \|Q\|_{L^q(\mathbb{T})} \ge$$

$$\geq \delta \left(\int_0^{1/N} |\ldots| \right)^{1/q} - C_q\sqrt{n} \geq \delta \left(\int_0^{1/N} \left| \frac{Nt}{\pi t} \right|^q \right)^{1/q} - C_q\sqrt{n} =$$

$$= \frac{n}{\pi N} N^{1-1/q} - C_q\sqrt{n} = \left(\frac{1}{\pi\sqrt{\varepsilon}} - C_q \right) \sqrt{n}.$$

Let us now consider $\|P\|_{L^q(\mathbb{T}\setminus E)}$ where E is an arc of measure $\nu < 1/2$ with the midpoint at 1. We have

$$\|P\|_{L^q(\mathbb{T}\setminus E)} \leq \left\| \delta \, \frac{z^N - 1}{z - 1} \right\|_{L^q(\mathbb{T}\setminus E)} + \|Q\|_{L^q(\mathbb{T}\setminus E)} \leq$$

$$\leq \delta \left(\frac{1}{1-\nu} \int_\nu^1 \left| \frac{e^{\pi i N t} - 1}{e^{\pi i t} - 1} \right|^q \right)^{1/q} + \left(\frac{1}{1-\nu} \right)^{1/q} \|Q\|_{L^q(\mathbb{T})} \leq$$

$$\leq 2\delta \left(\int_\nu^\infty \frac{1}{t^q} dt \right)^{1/q} + 2C_q\sqrt{n} = \frac{2n}{N} \, \nu^{\frac{1}{q}-1} + 2C_q \sqrt{n} \leq$$

$$\leq \frac{4n}{(\varepsilon n)^{q/2}} \, \nu^{\frac{1}{q}-1} + 2C_q\sqrt{n}, \text{ (if } n \text{ is large enough).}$$

If the pair (n, ν) is B-good we obtain from here

$$\frac{4n}{(\varepsilon n)^{q/2}} \, \nu^{\frac{1}{q}-1} \geq \frac{1}{B} \left(\frac{1}{\pi\sqrt{\varepsilon}} - (1+2B)C_q \right) \sqrt{n}.$$

If ε is small enough the right hand part is greater than $\sqrt{n}$, say, and therefore

$$\nu^{1-\frac{1}{q}} \leq 4\varepsilon^{-\frac{q}{2}} n^{\frac{1}{2}-\frac{q}{2}}.$$

Taking root of degree $1 - \frac{1}{q}$ of both sides we conclude that

$$\nu \leq \mathrm{Const}\, n^{-\frac{q}{2}}.$$

To construct an example in L^∞-case we put $N = [e^{\varepsilon n}]$.

So,

$$\|P\|_{L^\infty(\mathbb{T})} \geq \left\| \delta \, \frac{z^N - 1}{z-1} \right\|_{L^\infty(\mathbb{T})} - \|Q\|_{L^\infty(\mathbb{T})} \geq (1 - C_\infty \sqrt{\varepsilon}) n$$

while

$$\|P\|_{L^\infty(\mathbb{T}\setminus E)} \leq \left\| \delta \, \frac{z^N - 1}{z-1} \right\|_{L^\infty(\mathbb{T}\setminus E)} + \|Q\|_{L^\infty(\mathbb{T})} \leq \left(\frac{2}{N\nu} + C_\infty \sqrt{\varepsilon} \right) n.$$

Thus again

$$\frac{2}{N\nu} \geq \frac{1}{B} \left(1 - (B+1) C_\infty \sqrt{\varepsilon} \right)$$

and $\nu \leq \frac{\text{Const}}{N} \leq \text{Const}\, e^{-\varepsilon n}$, provided ε being chosen small enough.

It is worth mentioning that in all cases we have taken E to be an arc, though we are going to prove the result concerning arbitrary measurable sets ! Another interesting remark is that for polynomial of degree n the condition $\nu = O(1/n)$ is sufficient (and necessary, of course) for every $q \in [0, +\infty]$. Thus, there is no difference between behavior of polynomials of degree n and of order n in L^q-sense when $0 \leq q \leq 2$ and there is a great one when $q > 2$!

§3. Sufficiency.

First of all note that, as $\|p\|_{L^\infty(\mathbb{T})} \leq 2^{n-1} \|p\|_{L^0(\mathbb{T})}$ for any trigonometric polynomial of order n (we shall prove it below when we consider the L^0-case), we may not care of small values of n. Indeed, we have

$$\|p\|_{L^\infty(\mathbb{T})} \leq 2^{n-1} \|p\|_{L^0(\mathbb{T})} = 2^{n-1} \|p\|_{L^0(\mathbb{T}\setminus E)}^{1-\mu(E)} \|p\|_{L^0(E)}^{\mu(E)} \leq$$

$$\leq 2^{n-1} \|p\|_{L^0(\mathbb{T}\setminus E)}^{1-\mu(E)} \|p\|_{L^\infty(\mathbb{T})}^{\mu(E)} \text{ and thereby } \|p\|_{L^q(\mathbb{T})} \leq$$

$$\leq \|p\|_{L^\infty(\mathbb{T})} \leq 2^{\frac{n-1}{1-\mu(E)}} \|p\|_{L^0(\mathbb{T}\setminus E)} \leq 2^{\frac{n-1}{1-\mu(E)}} \|p\|_{L^q(\mathbb{T}\setminus E)}.$$

Thus any "finite" family of pairs $\{(n,\nu) : n \leq n_0,\ \nu \leq \nu_2 < 1\}$ is always good.

(This can be also concluded from some not very difficult purely topological considerations, but be cautious : only weak limits are allowed and (unlike the set of arcs of fixed length) the family of sets of measure ν is not compact in any sense).

A second useful remark is that instead of inequality $\|p\|_{L^q(\mathbb{T})} \le B\,\|p\|_{L^q(\mathbb{T}\setminus E)}$ we may check the inequality

$$\int\limits_E |p|^q\,d\mu \le B' \int\limits_{\mathbb{T}\setminus E} |p|^q\,d\mu.$$

(The cases $q = 0$ and $q = +\infty$ are, of course, exceptions of this rule).

§**3.1.** $\mathbf{2 < q < +\infty}$**.**

We start with this case because it is the simplest one (actually it is just trivial !).

Let $p(z) = \sum\limits_{k=1}^{n} c_k z^{m_k},\ E \subset \mathbb{T},\ \mu(E) \le \frac{A}{n^{q/2}}$.

Then

$$\int\limits_E |p|^q \le \mu(E)\left(\sum_{k=1}^{n}|c_k|\right)^q \le \mu(E)\left\{n\sum_{k=1}^{n}|c_k|^2\right\}^{q/2} =$$

$$= \mu(E)n^{q/2}\left(\sum_{k=1}^{n}|c_k|^2\right)^{q/2} \le A\left(\sum_{k=1}^{n}|c_k|^2\right)^{q/2}.$$

On the other side

$$\int\limits_{\mathbb{T}\setminus E}|p|^q \ge \left(\int\limits_{\mathbb{T}\setminus E}|p|^2\right)^{q/2} = \left(\int\limits_{\mathbb{T}}|p|^2 - \int\limits_{E}|p|^2\right)^{q/2} \ge$$

$$\ge \left\{\sum|c_k|^2 - \mu(E)(\sum|c_k|^2\right\}^{q/2} \ge \left(\sum|c_k|^2\right)^{q/2}(1-\mu(E)n)^{q/2} \ge$$

$$\ge \left(1 - A\,n^{1-q/2}\right)^{q/2}\left(\sum|c_k|^2\right)^{q/2} \ge \frac{1}{2}\left(\sum_{k=1}^{n}|c_k|^2\right)^{q/2}$$

if n is large enough. It remains only to compare two estimates obtained.

§**3.2.** $\mathbf{0 < q \le 2}$.

First of all let's note that if $\int\limits_E |p|^2 \le B' \int\limits_{\mathbb{T}\setminus E} |p|^2$ for every set E of measure $\mu(E) < \nu$ then $\int\limits_E |p|^q \le B' \int\limits_{\mathbb{T}\setminus E} |p|^q$ for every $0 < q \le 2$ and for every set $E \subset \mathbb{T}$ of measure $\mu(E) \le \nu$ as well. (Indeed, the worst case is when E consists of the points at which $|p|$ achieves its largest values, i.e. the case when $|p| \ge y$ on E and $|p| \le y$ outside E. But in this case $\int\limits_E |p|^q \le y^{q-2} \int\limits_E |p|^2 \le B' y^{q-2} \int\limits_{\mathbb{T}\setminus E} |p|^2 \le B' \int\limits_{\mathbb{T}\setminus E} |p|^q$).

This reduction can be done much more accurately. For example, I can prove that if the conjectured inequality $\|p\|_{L^q(\mathbb{T})} \le e^{A\mu(E)(n-1)} \|p\|_{L^q(\mathbb{T}\setminus E)}$ holds for $q = 2$, then it holds for each $q < 2$ (with, maybe, some greater constant A, not dependent on q nevertheless). A very similar thing was done in my previous article [1] in chapter 4, §2. If the reader is familiar with it, he will prove this new version without any difficulties (and even if not, he may succeed because it is quite easy, just too long to outline it here).

Let's continue now. We are going to show that for any $A > 0$ there exists some constant $B(A) > 0$ such that for every polynomial p of order n and any measurable subset $E \subset \mathbb{T}$ of measure $\mu(E) \le \min(1/2, \frac{A}{n})$ (minimum stands here just for formal reasons ; actually we are not interested in small values of n as was marked above) the inequality

$$\int\limits_E |p|^2 d\mu \le B(A) \int\limits_{\mathbb{T}\setminus E} |p|^2 d\mu$$

holds. Note that there is no problem when $A \le \frac{1}{2}$. Just write again $\int\limits_E |p|^2 d\mu \le$ $\le \mu(E) \left(\sum |c_k|\right)^2 \le \mu(E) n \sum\limits_{k=1}^{n} |c_k|^2 \le A \int\limits_{\mathbb{T}} |p|^2 d\mu$, and we may obviously take $B(A) = \frac{A}{1-A}$. To expand this simple estimate on large values of A we will use the "small shifts" technique developed in the third chapter of [1]. Being applied in its full strength it gives the estimate

$$\|p\|_{L^2(\mathbb{T})} \le e^{\mathrm{Const}\{n\mu(E)\}^{2+\varepsilon}} \|f\|_{L^2(\mathbb{T}\setminus E)}, \text{ when } \mu(E) < \frac{1}{2} < n\mu(E)$$

($\varepsilon > 0$ is arbitrarily small and the constant in the exponent depends on ε only). (That method works in the case of well-separated exponents as well). But I'm not going to prove this estimate here. I will outline only a very simple version of the "small shifts" method which is enough to solve the problem under consideration. It has some advantage nevertheless : you may use this version always, even when you have no shifts, just some not very small class of transformations for which you can control the situation. The simplest illustration is the following "uncertainty principle" for Fourier-Watson transforms :

Proposition.

Let $K:(0,+\infty)\to\mathbb{R}$ (or $\mathbb{C}$, it doesn't matter) be some kernel such that the operator $(Tf)(x)=\int\limits_0^\infty K(xy)f(y)dy$ acts in $L^2(0,+\infty)$ as a bounded linear operator, and T^{-1} exists and is a bounded operator as well.

If we suppose in addition that the decreasing rearrangement $K^*(t)$ of $|K|$ satisfies $\int\limits_0 K^*(t)^2\log\frac{1}{t}dt<+\infty^*$ then for each two measurable subsets $E,\Sigma\subset(0,+\infty)$ of finite measure the inequality

$$\|f\|^2_{L^2(0,+\infty)}\le C(|E||\Sigma|)\left\{\int\limits_{(0,+\infty)\setminus E}|f|^2+\int\limits_{(0,+\infty)\setminus\Sigma}|Tf|^2\right\}$$

holds, where $C\left(|E|\,|\Sigma|\right)$ is some positive constant, depending on the product of measures $|E|\,|\Sigma|$ only (and on K^*, of course).(For example, such kernels naturally appear when you take the Fourier transform for functions $f\in L^2(\mathbb{R}^n)$ depending on $|x|\ \ (x\in\mathbb{R}^n)$ only. In this case they are even bounded).

Let's continue, nevertheless. The main idea is to show that if $B\big((1-\delta)A\big)$ exists and δ is small enough (precise upper bound for δ will depend on A only), then $B(A)$ exists as well.

Indeed, let p be a polynomial of order n and E be some set of measure $\mu(E)\le\frac{A}{n}$.

Two cases are possible :

I) There exists a subset $E'\subset E$ of measure $\mu(E')=\delta\mu(E)$ and such that $\int\limits_{E'}|p|^2\le\frac{1}{2\big(B((1-\delta)A)+1\big)}\int\limits_E|p|^2$.

---

* This is merely the most natural sufficient condition to guarantee that the operator $M_ET^{-1}M_\Sigma TM_E$ ($M_Ef\overset{\text{def}}{=}f\chi_E$) is a Hilbert-Schmidt one (its Hilbert-Schmidt norm is obviously estimated from above by $\left\|T^{-1}\right\|\sqrt{\int\limits_0^{|E|\,|\Sigma|}K^*(t)^2\log\frac{|E|\,|\Sigma|}{t}\,dt}$).

Since $\mu(E\setminus E')=(1-\delta)\mu(E)\le\frac{(1-\delta)A}{n}$, we have

$$\int\limits_E |p|^2=\int\limits_{E\setminus E'}|p|^2+\int\limits_{E'}|p|^2\le B\big((1-\delta)A\big)\Big(\int\limits_{E'}|p|^2+\int\limits_{\mathbb{T}\setminus E}|p|^2\Big)+\int\limits_{E'}|p|^2\le$$

$$\le\big(B\big((1-\delta)A\big)+1\big)\int\limits_{E'}|p|^2+B\big((1-\delta)A\big)\int\limits_{\mathbb{T}\setminus E}|p|^2\le\frac{1}{2}\int\limits_E|p|^2+B\big((1-\delta)A\big)\int\limits_{\mathbb{T}\setminus E}|p|^2$$

and therefore $\int\limits_E|p|^2\le 2B\big((1-\delta)A\big)\int\limits_{\mathbb{T}\setminus E}|p|^2$ in this case.

II) For every subset $E'\subset E$ of measure $\delta\mu(E)$

$$\int\limits_{E'}|p|^2\ge\frac{1}{2\left(B\big((1-\delta)A\big)+1\right)}\int\limits_E|p|^2\overset{\text{def}}{=}\eta\int\limits_E|p|^2. \qquad (**)$$

Note that for any set $H\subset\mathbb{T}$ and for any trigonometric polynomial $p(z)=\sum\limits_{k=1}^n c_k z^{m_k}$ of order n we have

$$\int\limits_H|p|^2=\int\limits_{\mathbb{T}}\chi_H|p|^2=\sum_{i,j}c_i\overline{c}_j\int\limits_{\mathbb{T}}\chi_H z^{m_i-m_j}=\sum_{i,j}\widehat{\chi}_H(m_j-m_i)c_i\overline{c}_j=\langle Qp,p\rangle,$$

where Q is a self-adjoint operator in $L^2(\mathbb{T})$, matrix (q_{ij}) of which in standard basis $\{z^j\}_{j\in\mathbb{Z}}$ is defined by

$$\begin{cases} q_{m_jm_i} &= \widehat{\chi}_H(m_j-m_i)\\ q_{ij} &= 0 \text{ otherwise.}\end{cases}$$

Note that

$$\sum_{i,j}|q_{ij}|^2=\sum_{i=1}^n\sum_{j=1}^n|\widehat{\chi}_H(m_i-m_j)|^2\le\sum_{i=1}^n\sum_{m\in\mathbb{Z}}|\widehat{\chi}_H(m_i-m)|^2=n\mu(H)$$

and therefore Q is a Hilbert-Schmidt operator and $\|Q\|^2_{H-S}\le n\mu(H)$.

Let $\sigma_1,\sigma_2,\dots$ be the eigenvalues of Q enumerated in the absolute value nonincreasing order, $V_k(H)$ be the k-dimensional subspace of $L^2(\mathbb{T})$ containing eigenvectors, corresponding to $\sigma_1,\dots,\sigma_k$.

For any polynomial p orthogonal to $V_d(H)$ we have $\|Q\|_{L^2(\mathbb{T})} \le |\sigma_{d+1}|\ \|p\|_{L^2(\mathbb{T})}$. Thus, if $|\sigma_{d+1}| \le \frac{1}{2}$ then $\langle Qp, p\rangle \le \frac{1}{2}\|p\|^2_{L^2(\mathbb{T})}$ and thereby $\int\limits_H |p|^2 \le \int\limits_{\mathbb{T}\setminus H} |p|^2$. But

$$|\sigma_{d+1}|^2 \le \frac{1}{d+1}\sum_{j=1}^{\infty}|\sigma_j|^2 = \frac{1}{d+1}\sum_{i,j}|q_{ij}|^2 \le \frac{n\mu(H)}{d+1}$$

and therefore we may put $d = [4n\mu(H)]$ to provide it. Now we are able to describe in few words the way we are going to follow.

We shall consider several shifts of p : $\tau_{\zeta_0}p,\ \tau_{\zeta_1}p, \ldots, \tau_{\zeta_d}p$, say. (We put$(\tau_\zeta p)(z) \stackrel{\text{def}}{=} p(\frac{z}{\zeta})$, $\zeta \in \mathbb{T}$; these shifts are, obviously, trigonometric polynomials with the same spectrum as p). We choose these shifts in such a way that $\zeta_0 = 1$ (so, $\tau_{\zeta_0}p = p$) and $\mu(\zeta_1 E\setminus E) = \mu\big(\zeta_2 E\setminus(\zeta_1 E\cup E)\big) = \ldots = \mu(\zeta_d E\setminus \bigcup\limits_{j=0}^{d-1}\zeta_j E) = \delta\mu(E)$ (and thereby $\mu\Big(\bigcup\limits_{j=0}^{n}\tau_{\zeta_j}E\Big) \stackrel{\text{def}}{=} \mu(H) = (1+\delta d)\mu(E)$). If $\delta < \frac{1}{d}$ then $\mu(H) \le 2\mu(E)$ and $n\mu(H) \le 2A$. Thus, if $d+1 > 8A$ (for example, if $d = [8A]$) we can find some linear combination

$$g = a_0 p + a_1\tau_{\zeta_1}p + \ldots + a_d\tau_{\zeta_d}p \quad \Big(a_j \in \mathbb{C},\ \sum_j |a_j|^2 = 1\Big)$$

which is orthogonal to $V_d(H)$ and therefore satisfies

$$\int\limits_H |g|^2 \le \int\limits_{\mathbb{T}\setminus H}|g|^2 = \int\limits_{\mathbb{T}\setminus H}\left|\sum_{j=0}^{d} a_j\tau_{\zeta_j}p\right|^2 \le \int\limits_{\mathbb{T}\setminus H}\Big(\sum_j |a_j|^2\Big)\sum_j|\tau_{\zeta_j}p|^2 =$$

$$= \sum_{j=0}^{d}\int\limits_{\mathbb{T}\setminus H}|\tau_{\zeta_j}p|^2 \le \sum_{j=0}^{d}\int\limits_{\mathbb{T}\setminus\zeta_j E}|\tau_{\zeta_j}p|^2 = (d+1)\int\limits_{\mathbb{T}\setminus E}|p|^2.$$

The condition $(**)$ will allow us to estimate $\int\limits_H |g|^2$ from below by $\eta'\int\limits_E |p|^2 - d\int\limits_{\mathbb{T}\setminus E}|p|^2$ with some small positive constant η', depending on η, d and $B\big((1-\delta)A\big)$ only, and we shall get the inequality $\int\limits_E |p|^2 \le \frac{2d+1}{\eta'}\int\limits_{\mathbb{T}\setminus E}|p|^2$ which is just what we need.

Now let's make precise calculations.

First of all : why are we sure that the shifts needed do really exist ? We shall prove that it is so if $\delta \le \frac{1}{2d} \le \frac{1}{2}$ and $\mu(E) < \frac{1}{4}$ (it can be assumed without loss of generality because $\mu(E) \le \frac{A}{n}$ and we are not interested in small values of n).

Indeed, the shifts $\tau_{\zeta_1}, \dots, \tau_{\zeta_k}$ being already constructed, consider the measure $\mu(\tau_\zeta E \setminus \bigcup\limits_0^k \zeta_j E)$ as a function of ζ. It is continuous, equals 0 at $\zeta = 1$ and its integral over $\mathbb{T}$ is equal to

$$\mu(E)\,\Big(1 - \mu\big(\bigcup_0^k \zeta_j E\big)\Big) = \mu(E)\big(1-(1+\delta k)\mu(E)\big) \ge \frac{1}{2}\,\mu(E) > \delta\mu(E) \ \text{ if } \ k<d.$$

Thus one can find some point ζ_{k+1} at which this function equals $\delta\mu(E)$. It remains only to do it successively for $k = 1, \dots, d-1$.

Now, how can $(**)$ help us to estimate $\int\limits_H |g|^2$ from below ? For every $k \in \{0, 1, \dots, d\}$ we have

$$\int\limits_H |g|^2 \ge \int\limits_{\zeta_k E \setminus \left(\bigcup\limits_{j=0}^{k-1} \zeta_j E\right)} |g|^2 = \int\limits_{\dots} \left| \sum_0^d a_j \tau_{\zeta_j} p \right|^2 \ge$$

$$\ge \frac{1}{3} \int\limits_{\dots} |a_k \tau_{\zeta_k} p|^2 - \int\limits_{\dots} \left| \sum_{j<k} a_j \tau_{\zeta_j} p \right|^2 - \int\limits_{\dots} \left| \sum_{j>k} a_j \tau_{\zeta_j} p \right|^2 .$$

(We have used the inequality $|x+y+z|^2 \ge \frac{1}{3}|x|^2 - |y|^2 - |z|^2$ which is nothing but another form of $|(x+y+z)-y-z|^2 \le 3\big(|x+y+z|^2 + |y|^2 + |z|^2\big)$.)

Let's look at three terms obtained more attentively.

$$\int\limits_{\zeta_k E \setminus \left(\bigcup\limits_{j=0}^{k-1} \zeta_j E\right)} |a_k \tau_{\zeta_k} p|^2 = |a_k|^2 \int\limits_{E \setminus \zeta_k^{-1} \bigcup\limits_{j=0}^{k-1} \zeta_j E} |p|^2.$$

But the set, the integral is taken over, is the same subset of E of measure $\delta\mu(E)$ (or even $\mu(E)$ when $k=0$) and therefore according to $(**)$ the integral itself is not less than $\eta |a_k|^2 \int\limits_E |p|^2$.

The second term can be estimated from above by

$$\Big(\sum_{j<k}|a_j|^2\Big)\sum_{j<k}\int\limits_{\zeta_k E\setminus\bigcup\limits_0^{k-1}\zeta_j E}\left|\tau_{\zeta_j}p\right|^2\le\sum_{j<k}\int\limits_{\mathbb{T}\setminus\zeta_j E}\left|\tau_{\zeta_j}p\right|^2=k\int\limits_{\mathbb{T}\setminus E}|p|^2.$$

At last, the third term doesn't exceed

$$\int\limits_{\mathbb{T}}\Big|\sum_{j>k}a_j\tau_{\zeta_j}p\Big|^2\le\Big(\sum_{j>k}|a_j|^2\Big)\sum_{j>k}\int\limits_{\mathbb{T}}\left|\tau_{\zeta_j}p\right|^2=$$

$$=\sum_{j>k}|a_j|^2\,(d-k)\int\limits_{\mathbb{T}}|p|^2\le\sum_{j>k}|a_j|^2\,d\int\limits_{E}|p|^2+(d-k)\int\limits_{\mathbb{T}\setminus E}|p|^2.$$

So, we have

$$\int\limits_{H}|g|^2\ge\Big(\frac{\eta}{3}\,|a_k|^2-d\sum_{j>k}|a_j|^2\Big)\int\limits_{E}|p|^2-d\int\limits_{\mathbb{T}\setminus E}|p|^2$$

for each k and thereby

$$\int\limits_{H}|g|^2\ge\max_{k=0}^{d}\Big(\frac{\eta}{3}\,|a_k|^2-d\sum_{j>k}|a_j|^2\Big)\int\limits_{E}|p|^2-d\int\limits_{\mathbb{T}\setminus E}|p|^2.$$

Note that the factor before $\int\limits_E|p|^2$ is always positive unless $a_0=a_1=\ldots=a_d=0$ and, obviously, continuous in a_j. Thus it is greater than some positive constant η', say, on the sphere $|a_0|^2+\ldots+|a_d|^2=1$, just as we were going to show.

We now have proved that for $A>1/2$ the constant $B(A)$ exists, provided that $B\big(A(1-\frac{1}{2[8A]})\big)$ does. It remains only to note that the sequence defined by $A_0=A$, $A_{k+1}=A_k\big(1-\frac{1}{2[8A_k]}\big)$ will always abrupt at some term $A_s<1/2$ and therefore $B(A)$ exists for every $A>0$.

§3.3 $\mathbf{q=0}$.

We shall use the Bernstein inequality $\left\|\frac{d}{dz}\,p\right\|_{L^q(\mathbb{T})}\le M\,\|p\|_{L^q(\mathbb{T})}$ (where $p=\sum\limits_{-M}^{M}c_m z^m$) in the same way as we used a pointwise estimate of weak type in [1]. We need only the case of L^0-norm which is well-known for about 30 years (since Maler's work). So, I omit the proof. The reader may find it in [2].*

First of all let's prove the promised estimate $\|p\|_{L^\infty(\mathbb{T})} \le 2^{n-1}\|p\|_{L^0(\mathbb{T})}$ for any polynomial of order n (as the example $p(z) = (1+z)^{n-1}$ shows, it is the best possible one). Let $p(z) = \sum\limits_{j=1}^{n} c_j z^{m_j}$. We shall even show that $\|p\|_W = \sum\limits_{j=1}^{n} |c_j| \le 2^{n-1}\|p\|_{L^0(\mathbb{T})}$.

Consider two polynomials : $\overline{p}(z) \overset{\text{def}}{=} \frac{d}{dz}\left(z^{-m_1}p(z)\right)$ and $\underline{p}(z) \overset{\text{def}}{=} \frac{d}{dz}\left(z^{-m_n}p(z)\right)$. Both of them are, obviously, trigonometric polynomials of order $n-1$. As $z^{-m_1}p$ and $z^{-m_n}p$ are trigonometric polynomials of degree $m_n - m_1 \overset{\text{def}}{=} m$, we get

$$\|\overline{p}\|_{L^0(\mathbb{T})} \le m\left\|z^{-m_1}p\right\|_{L^0(\mathbb{T})} = m\|p\|_{L^0(\mathbb{T})}$$

and

$$\left\|\underline{p}\right\|_{L^0(\mathbb{T})} \le m\|p\|_{L^0(\mathbb{T})}$$

as well.

Note that $z^{m_1+1}\overline{p} - z^{m_n+1}\underline{p} = (m_n - m_1)p = mp$ and therefore $\|\overline{p}\|_W + \left\|\underline{p}\right\|_W \ge m\|p\|_W$.

Assume that $\|\overline{p}\|_W \ge \frac{m}{2}\|p\|_W$ (otherwise we are to take $\underline{p}$). Then the polynomial $\tilde{p} = \frac{1}{m}\overline{p}$ satisfies $\operatorname{ord}\tilde{p} = n-1$, $\|\tilde{p}\|_W \ge \frac{1}{2}\|p\|_W$ and $\|\tilde{p}\|_{L^0(\mathbb{T})} \le \|p\|_{L^0(\mathbb{T})}$. Iterating this construction $n-1$ times, we obtain a sequence of polynomials $p_n, \dots, p_1$ where $p_n = p$, $\operatorname{ord} p_k = k$, $\|p_{k+1}\|_W \ge \frac{1}{2}\|p_k\|_W$ and, at last, $\|p_{k+1}\|_{L_0(\mathbb{T})} \le \|p_k\|_{L^0(\mathbb{T})}$. So we have

$$\|p\|_W = \|p_n\|_W \le 2^{n-1}\|p_1\|_W = 2^{n-1}\|p_1\|_{L^0(\mathbb{T})} \le 2^{n-1}\|p\|_{L^0(\mathbb{T})}\,.$$

(It is easy to show also (by induction in n) that the equality holds iff $p(z) = \lambda z^m(1+\mu z^r)^{n-1}$ ($\lambda \in \mathbb{C}, \mu \in \mathbb{T}$)). Now we are able to prove sufficiency of $(*)$: just write

$$\|p\|_{L^0(\mathbb{T})} = \|p\|_{L^0(E)}^{\mu(E)}\|p\|_{L^0(\mathbb{T}\setminus E)}^{1-\mu(E)} \le \|p\|_{L^\infty(\mathbb{T})}^{\mu(E)}\|p\|_{L^0(\mathbb{T}\setminus E)}^{1-\mu(E)}$$

$$\le 2^{(n-1)\mu(E)}\|p\|_{L^0(\mathbb{T})}^{\mu(E)}\|p\|_{L^0(\mathbb{T}\setminus E)}^{1-\mu(E)}$$

* together with a very simple and beautiful trick which allows to reduce the case of any L^q-norm with $q>0$ to the case of L^0-norm and a long discussion of what is the most general case when this technique works.

and thereby

$$\|p\|_{L^0(\mathbb{T})} \le 2^{\frac{\mu(E)}{1-\mu(E)}(n-1)} \|p\|_{L^0(\mathbb{T}\setminus E)} \le e^{2\mu(E)(n-1)} \|p\|_{L^0(\mathbb{T}\setminus E)}$$

when $\mu(E) \le \frac{1}{2}$.

Using the results of [1] it is very easy to show that the inequality $\|p\|_{L^\infty(I)} \le e^{A(n-1)} \|p\|_{L^0(I)}$ holds for every exponential polynomial with purely imaginary exponents (no separation condition is needed !) and therefore we may think the case $q = 0$ to be investigated in its full generality.

§**3.4 q** $= +\infty$.

To begin with let's prove the statement for the case when all c_j are real positive numbers (and therefore $\|p\|_{L^\infty(\mathbb{T})} = \sum_{j=1}^{n} c_j$). We shall not use this partial result in general case but it is too simple and illustrates too useful ideas to be neglected.

Consider together with the polynomial p the sum $g = \sum_{k=1}^{n} c_k e^{2\pi i \xi_k}$ where ξ_k are independent random variables equidistributed on $[-1/2, 1/2]$.

Fix some positive integer r. We have

$$\int_{\mathbb{T}} |p|^{2r} = \sum_{\substack{m_{i_1}+\ldots+m_{i_r}=m_{j_1}+\ldots+m_{j_r} \\ 1\le i_1,\ldots,i_r,j_1,\ldots,j_r \le n}} c_{i_1}\ldots c_{i_r} c_{j_1}\ldots c_{j_r} \ge$$

$$\sum_{\substack{\{i_1,\ldots,i_r\}=\{j_1\ldots j_r\} \\ \text{counting with multiplicity}}} c_{i_1}\ldots c_{i_r} c_{j_1}\ldots c_{j_r}$$

for all terms are positive. (The words "counting with multiplicity" mean that $\{1,2,1,3\} = \{3,2,1,1\}$, but $\{1,2,1\} \ne \{1,2,2\}$). The latter sum is precisely an expression for $\int_{\mathbb{T}} |g|^{2r}$. So, we have $\int_{\mathbb{T}} |p|^{2r} \ge \int_{\Omega} |g|^{2r}$. Fix $\varepsilon \in (0, \frac{1}{8})$. Let $\Omega' = \{\omega : |\xi_k(\omega)| \le \frac{1-\varepsilon}{2}\ (k = 1, \ldots, n)\} \subset \Omega$. Obviously $P(\Omega') = (1-\varepsilon)^n$ and $\frac{1}{P\{\Omega'\}} \int_{\Omega'} \xi_j = \frac{\sin \pi\varepsilon}{\pi(1-\varepsilon)} \ge \frac{\varepsilon}{2}$, say. Thus

$$\frac{1}{P\{\Omega'\}} \int_{\Omega} |g|^{2r} \ge \frac{1}{P\{\Omega'\}} \int_{\Omega'} |g|^{2r} \ge \left\{ \frac{1}{P\{\Omega'\}} \left| \int_{\Omega'} g \right| \right\}^{2r} \ge \Big(\frac{\varepsilon}{2} \sum_{1}^{n} c_j\Big)^{2r},$$

and

$$\int_{\mathbb{T}} |p|^{2r} \geq (1-\varepsilon)^n \left(\frac{\varepsilon}{2}\right)^{2r} \left(\sum_{j=1}^{n} c_j\right)^{2r} \qquad \text{for every } r \in \mathbb{N}.$$

Suppose now that $|p| \leq \frac{1}{B} \sum\limits_{j=1}^{n} c_j$ outside some set $E \subset \mathbb{T}$ of measure $\nu \leq e^{-4\varepsilon n}$.

Then

$$\left(B^{-2r} + \nu\right) \left(\sum c_j\right)^{2r} \geq \int_{\mathbb{T}} |p|^{2r} \geq e^{-2\varepsilon n} \left(\frac{\varepsilon}{2}\right)^{2r} \left(\sum_{j=1}^{n} c_j\right)^{2r} \text{ (for } 1-\varepsilon \geq e^{-2\varepsilon}\text{)}$$

and therefore

$$B^{-2r} \geq e^{-2\varepsilon n} \left(\frac{\varepsilon}{2}\right)^{2r} - e^{-4\varepsilon n} = e^{-(2\varepsilon n + 2r \log(2/\varepsilon))} - e^{-4\varepsilon n}.$$

Put $r = \left[\frac{\varepsilon}{2\log(2/\varepsilon)} n\right] \geq \frac{\varepsilon n}{4\log(2/\varepsilon)}$, provided n being large enough. Then $B^{-2r} \geq e^{-3\varepsilon n} - e^{-4\varepsilon n} \geq e^{-4\varepsilon n}$ if n is not very small and $B \leq e^{\frac{2\varepsilon n}{r}} \leq \left(\frac{2}{\varepsilon}\right)^{8}$, proving the case of positive coefficients.

Now let's consider the general case.

Let E be a small subset of $\mathbb{T}$ of measure $\mu(E) \leq e^{-\varepsilon n}$. We are to show that $\sup\limits_{\mathbb{T}} |p| \leq B \sup\limits_{\mathbb{T}\setminus E} |p|$.

The polynomial p achieves its maximum at some point $z_0 \in \mathbb{T}$. One can find such $\zeta_0 \in \mathbb{T}$ that for any nonzero integer $j \in \left(-e^{\frac{\varepsilon n}{2}}, e^{\frac{\varepsilon n}{2}}\right)$ the point $z_0 \zeta_0^j \notin E$.

Indeed, for every fixed j the measure of the set $\{\zeta \in \mathbb{T} : \zeta^j z_0 \in E\}$ is equal to $\mu(E) < e^{-\varepsilon n}$. Thus the measure of the union of these sets is less than $2e^{\varepsilon n/2} e^{-\varepsilon n} = 2e^{-\varepsilon n/2} < 1$ provided n being large.

Now we are going to apply the well-known Turan idea and construct a trigonometric polynomial $R(\zeta) = \sum\limits_{|j| \leq e^{\varepsilon n/2}} \widehat{R}_j \zeta^j$ such that

1) $\widehat{R}_0 = 0$ (or, by other words, $\int\limits_{\mathbb{T}} R d\mu = 0$.)

2) $|R(\zeta_0^{m_k}) - 1| < \frac{1}{2n}$ for each $k = 1, \ldots, n$.

3) $\sum\limits_j \left|\widehat{R}_j\right| \le \frac{\text{Const}}{\varepsilon}$.

If we had got such a polynomial we could write

$$\|p\|_{L^\infty(\mathbb{T})} = |p(z_0)| = \left|\sum_{k=1}^{n} c_k z_0^{m_k}\right| \le \left|\sum_{k=1}^{n} c_k z_0^{m_k} R(\zeta_0^{m_k})\right| +$$

$$+\sum_{k=1}^{n} |c_k|\,|1 - R(\zeta_0^{m_k})| \le \left|\sum_j \widehat{R}_j \Big(\sum_k c_k (z_0\zeta_0^j)^{m_k}\Big)\right| + \frac{1}{2n}\sum_{k=1}^{n}|c_k|$$

$$\le \sum_j \left|\widehat{R}_j\right| \left|p(z_0\zeta_0^j)\right| + \frac{1}{2n}\sum_{k=1}^{n}|c_k| \le$$

$$\le \sum_j \left|\widehat{R}_j\right| \sup_{\mathbb{T}\setminus E}|p| + \frac{1}{2n}\sum_{k=1}^{n}|c_k|.$$

Note that at any rate $\|p\|_{L^\infty(\mathbb{T})} \ge \max\limits_{k=1}^{n} |c_k| \ge 2\big(\frac{1}{2n}\sum\limits_k |c_k|\big)$ and therefore $\|p\|_{L^\infty(\mathbb{T})} \le 2\sum\limits_j \left|\widehat{R}_j\right| \|p\|_{L^\infty(\mathbb{T}\setminus E)} \le \frac{\text{Const}}{\varepsilon} \|p\|_{L^\infty(\mathbb{T}\setminus E)}$. It remains only to find a polynomial R with the desired properties. Denote $\zeta_k \stackrel{\text{def}}{=} \zeta_0^{m_k}$ for convenience. According to the Dirichlet theorem on simultaneous rational approximation we can choose some integer $d \in [0, e^{\varepsilon n/4}]$ such that $\left|\arg \zeta_k^d\right| \le e^{-\varepsilon/4}\pi$ for all k (i.e. none of the numbers ζ_k^d lies on the arc $\mathcal{L} \subset \mathbb{T}$ of measure $1 - e^{-\varepsilon/4}$ centred at -1). Consider the function $\varphi(\zeta)$ which equals 1 on $\mathbb{T}\setminus\mathcal{L}$, equals $-\frac{1+e^{-\varepsilon/4}}{1-e^{-\varepsilon/4}}$ at -1 and is linear (in $\arg\zeta$) on each half of the arc $\mathcal{L}$. It is easy to check that $\int\limits_{\mathbb{T}} \varphi = 0$, $\sum\limits_{j\in\mathbb{Z}} |\widehat{\varphi}_j| = \frac{1+e^{-\varepsilon/4}}{1-e^{\varepsilon/4}} \le \frac{16}{\varepsilon}$ and $|\varphi_j| \le \frac{100}{\varepsilon^2 j^2}$, say. Now it is evident that we may put $R(\zeta) = S_{[e^{\varepsilon n/4}]}\varphi(\zeta^d)$ for large n, where $S_m\varphi \stackrel{\text{def}}{=} \sum\limits_{-m}^{m} \widehat{\varphi}_j \zeta^j$ is the m-th partial sum of the Fourier series of φ.

Very similar arguments allow us to show that for large ε the ratio of norms $\|p\|_{L^\infty(\mathbb{T})}$ and $\|p\|_{L^\infty(\mathbb{T}\setminus E)}$ is close to 1 (for other norms the corresponding statement is obvious if you look at the proofs attentively). Actually you get something like

$$\|p\|_{L^\infty(\mathbb{T})} \le \left\{1 + \mathrm{Const}\,\mu(E)^{\frac{1}{2(n-1)}}\right\} \|p\|_{L^\infty(\mathbb{T}\setminus E)}$$

for all $\mu(E) \le e^{-n}$, say; while the example of independent random variables equidistributed on $\mathbb{T}$ shows that the ratio may be greater than $1 + \mathrm{const}\,\mu(E)^{\frac{2}{n-1}}$ in the same area.

To get the analogue of this result for well-separated exponents, it is enough to construct an algebraic polynomial $R(\zeta)$ with the same properties instead of a trigonometric one but I have failed in doing it. It is also possible to prove an analogue of this result for an interval without any separation condition imposed on exponents, but under the assumption that the maximum value is achieved in the "middle half" of the interval. As I firmly believe that this assumption is quite superfluous, I think this result to be of very small value and don't outline its proof here.