**A counterexample to Sarason's conjecture**

## §0. Introduction.

Consider the Hilbert transform operator $H$ defined (on not too bad functions $\varphi$) by

$$H\varphi(t) := \frac{1}{\pi}\,\text{v. p.}\int_{\mathbb{R}} \frac{\varphi(s)ds}{t-s}.$$

It is well-known that $H$ is a unitary operator in $L^2$. A very interesting and still unsolved question is how to describe all pairs of "weights" (nonnegative functions) $w, v : \mathbb{R} \to [0, +\infty)$ such that the weighted Hilbert transform operator

$$\widetilde{H} := v^{\frac{1}{2}} H w^{\frac{1}{2}}$$

acts in $L^2$ (if $w$ does not vanish at any point of the real line, this is equivalent to the action of $H$ from $L^2(w^{-1})$ to $L^2(v)$).

In the "one-weight" case ($w = v^{-1}$ in our notation), the answer is given by the celebrated

**Hunt-Muckenhoupt-Weeden theorem:**

The weighted Hilbert transform $\widetilde{H} := v^{\frac{1}{2}} H v^{-\frac{1}{2}}$ acts in $L^2$ if and only if there is some constant $C > 0$ such that for every interval $I \subset \mathbb{R}$, one has

$$\langle v\rangle_I \langle v^{-1}\rangle_I \le C.$$

(For a function $\varphi : \mathbb{R} \to \mathbb{C}$ and an interval $I \subset \mathbb{R}$, we will always denote $\langle\varphi\rangle_I := \frac{1}{|I|}\int_I \varphi(t)\,dt$. Also, "interval" will always mean "closed interval" unless otherwise specified).

The corresponding "two-weight" analogue

$$\langle v\rangle_I \langle w\rangle_I \le C \qquad (*)$$

of this condition is necessary in the general case as well, but, unfortunately, no longer sufficient. Indeed, put

$$w(t) := \chi_{[0,1]}(t), \qquad v(t) := |t|, \qquad \varphi(t) := \chi_{[0,1]}(t)$$

(here and later on, we denote by $\chi_E$ the characteristic function of a set $E \subset \mathbb{R}$).

It is easy to check that $(*)$ is satisfied with $C = 2$, say. Also, clearly, $\varphi \in L^2$ and $\|\varphi\|_{L^2} = 1$. On the other hand $\widetilde{H}\varphi(t) \approx \frac{1}{\pi\sqrt{t}}$ as $t \to +\infty$ and therefore $\widetilde{H}\varphi \notin L^2$.

This (very old and well-known) counterexample is based on the fact that the Hilbert transform is a "far-acting" operator (i.e., its kernel decays slowly at infinity), while the condition $(*)$ is, in a sense, a "local" condition. The idea of Sarason was to go around

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the difficulty by replacing the Steklov averages by the Poisson ones. Recall that for any non-negative function $\psi$ on the real line, we can define its Poisson extension to the upper half-plane $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ by

$$P\psi(z) := \frac{1}{\pi}\int_{\mathbb{R}} \frac{\operatorname{Im} z}{|z-t|^2}\psi(t)\,dt.$$

Sarason conjectured that the condition

$$Pv(z)\cdot Pw(z) \le C \qquad \text{for every } z \in \mathbb{C}_+ \qquad (**)$$

would be necessary and sufficient for the action of $\widetilde{H}$ in $L^2$.

Note that this condition is stronger than $(*)$, because for every interval $I \subset \mathbb{R}$ and for every non-negative function $\psi$, $P\psi(z_I) \ge \pi^{-1}\langle\psi\rangle_I$, where $z_I$ is defined as the point for which $\operatorname{Re} z_I$ is the center of $I$ and $\operatorname{Im} z_I = \frac{|I|}{2}$.

Note also that in the example above, the Poisson extension $Pw$ is some decent everywhere positive function, while the Poisson extension of $v$ blows up: $Pv \equiv +\infty$ everywhere in $\mathbb{C}_+$. So, the Sarason condition $(**)$ rules out this simple counterexample (and many other counterexamples of the same kind). What is more, almost immediately after the conjecture appeared, Sergei Treil proved that $(**)$ is a necessary condition.

It is a pity that despite all these attractive features (so attractive that they made a lot of people believe into validity of what, as Sarason himself once confessed, appeared "just as a wild guess"), the Sarason conjecture is false. And actually it is much worse than just false: it is *misleading* in the sense that, as we shall see below, no slight modification of the Muckenhoupt condition can be necessary and sufficient simultaneously. It is probably worth mentioning here that there exists a very nice "slight modification" of the condition $(*)$ that is *sufficient* (but by no means necessary): I mean the so-called Dechao Zheng condition

$$\langle v^\beta\rangle_I \langle w^\beta\rangle_I \le C$$

where $\beta$ is any number strictly greater than 1. (Note that in the "one-weight" case this improvement of power comes for free because of the inverse Holder inequalities; so Dechao Zheng's theorem is formally stronger than the Hunt-Muckenhoupt-Weeden that).

## §1. $\delta$-Smooth weights.

In order to construct a counterexample to Sarason's conjecture, let us observe that there is one important special case when the conditions $(*)$ and $(**)$ coincide. Namely, it is so if both the weights $v$ and $w$ are "$\delta$-smooth" for some small $\delta > 0$, i.e., for every interval $I \subset \mathbb{R}$,

$$\frac{\langle w\rangle_{I_+}}{\langle w\rangle_{I_-}} \in (e^{-\delta}, e^{\delta}),$$

and the same for $v$. (From now on, we will always denote by $I_-$ and $I_+$ the left and the right halves of an interval $I \subset \mathbb{R}$ respectively). Indeed, an immediate consequence of $\delta$-smoothness is that

$$\langle w\rangle_{2I} \le e^{\delta}\langle w\rangle_I,$$

where $2I$ is the interval centered at the same point as $I$, but twice longer.

Now for any point $z \in \mathbb{C}_+$, we can take $I := [\operatorname{Re} z - \operatorname{Im} z, \operatorname{Re} z + \operatorname{Im} z]$ and write

$$\pi Pw(z) = \int_{\mathbb{R}} \frac{\operatorname{Im} z}{|t-z|^2} w(t)\, dt = \int_I \cdots + \sum_{k=1}^{\infty} \int_{2^k I \setminus 2^{k-1} I} \cdots$$
$$\le 2\Big[\langle w\rangle_I + \sum_{k=1}^{\infty} 2^{-k} \langle w\rangle_{2^k I}\Big] \le 2\Big(1 + \sum_{k=1}^{\infty} 2^{-k} e^{k\delta}\Big) \langle w\rangle_I =: \pi A_\delta \langle w\rangle_I,$$

and the same for $v$.

So, if the product $\langle v\rangle_I \langle w\rangle_I$ is bounded by some constant $C$ for every interval $I \subset \mathbb{R}$, then the product $Pv(z)Pw(z)$ is bounded by $A_\delta^2 C$ for every $z \in \mathbb{C}_+$ (and $A_\delta < +\infty$ for any $\delta < \log 2$).

We will demonstrate that for every (small) $\delta > 0$ and for every (large) $M > 0$, one can find a pair of $\delta$-smooth weights $v, w$ satisfying

$$\langle v\rangle_I \langle w\rangle_I \le 1 \qquad \text{for every interval } I \subset \mathbb{R}$$

such that

$$\|\widetilde{H}\|_{L^2 \to L^2} \ge M.$$

This shows that at least one cannot have an effective estimate for the norm of the weighted Hilbert transform in the Sarason conjecture. To improve our construction to an example, in which the weighted Hilbert transform is unbounded, is left to the reader as an exercise.

Note also that the Poisson kernel $\frac{\operatorname{Im} z}{|t-z|^2}$ in this reasoning can be replaced by much slower decaying kernel $\frac{(\operatorname{Im} z)^{2\delta}}{|t-z|^{1+2\delta}}$. The corresponding condition for general weights (not satisfying any a priori smoothness assumption) will be no longer necessary (if $\delta < \frac{\log 2}{2}$), but still not sufficient.

## §2. The dyadic model: Haar multipliers.

Let $\mathcal{D}$ be a dyadic lattice. With every dyadic interval $I \in \mathcal{D}$, one can associate the corresponding Haar function

$$h_I(t) := \begin{cases} -1, & \text{if } t \in I_-; \\ +1, & \text{if } t \in I_+; \\ 0, & \text{if } t \notin I. \end{cases}$$

It is well-known and easy to check that Haar functions form an orthogonal basis in $L^2$. Hence, every function $\varphi \in L^2$ can be decomposed into its Haar series

$$\varphi = \sum_{I \in \mathcal{D}} a_I h_I,$$

where

$$a_I = \frac{(\varphi, h_I)_{L^2}}{\|h_I\|^2_{L^2}} = \frac{1}{|I|}\Big(\int_{I_+} \varphi(t)\,dt - \int_{I_-} \varphi(t)\,dt\Big) = \frac{1}{2}(\langle\varphi\rangle_{I_+} - \langle\varphi\rangle_{I_-}) =: \Delta_I\varphi.$$

Let $\varepsilon = \{\varepsilon_I\}_{I\in\mathcal{D}}$ be a sequence of signs (i.e., $\varepsilon_I = \pm 1$). Define the operators $T_\varepsilon$ (Haar multipliers) by

$$T_\varepsilon h_I := \varepsilon_I h_I$$

and the weighted operators $\widetilde{T}_\varepsilon$ by $\widetilde{T}_\varepsilon := v^{\frac12} T_\varepsilon w^{\frac12}$. All the operators $T_\varepsilon$ are unitary Calderon-Zygmund operators in $L^2$. So, in a sense, they may serve as a model for the Hilbert transform operator. Our first aim will be to show that for any (large) $M > 0$ and for any (small) $\delta > 0$, there exist two weights $v$ and $w$ such that for every $I \in \mathcal{D}$

$$\langle v\rangle_I\langle w\rangle_I \le 1 \qquad \text{and} \qquad \frac{\langle v\rangle_{I_+}}{\langle v\rangle_{I_-}}, \frac{\langle w\rangle_{I_+}}{\langle w\rangle_{I_-}} \in (e^{-\delta}, e^{\delta}) \qquad (*dy)$$

but the norms of the weighted operators $\widetilde{T}_\varepsilon$ are greater than $M$ in average (see the exact formulation below).

Note that the dyadic versions of the Muckenhoupt condition and the $\delta$-smoothness condition do not imply their full versions (if something is true for each interval in a fixed dyadic lattice, it doesn't mean that it is true for every interval on the real line).

Let $I \in \mathcal{D}$ be any dyadic interval, let $v, w$ be any pair of weights and, at last, let $\varphi \in L^2$. We have

$$\|\widetilde{T}_\varepsilon\varphi\|^2_{L^2} = \int_{\mathbb{R}} \Big|\sum_{J\in\mathcal{D}} \varepsilon_J \Delta_J(\varphi w^{\frac12}) h_J(t)\Big|^2 v(t)\,dt.$$

Taking the average over all possible choices of signs, we get

$$\begin{aligned}\mathcal{E}_\varepsilon\|\widetilde{T}_\varepsilon\varphi\|^2_{L^2} &= \int_{\mathbb{R}} \sum_{J\in\mathcal{D}} |\Delta_J(\varphi w^{\frac12}) h_J(t)|^2 v(t)\,dt \\ &= \sum_{J\in\mathcal{D}} |\Delta_J(\varphi w^{\frac12})|^2 \langle v\rangle_J |J| \ge \sum_{J\in\mathcal{D}, J\subset I} |\Delta_J(\varphi w^{\frac12})|^2 \langle v\rangle_J |J|.\end{aligned}$$

If the norms of the operators $\widetilde{T}_\varepsilon$ are "bounded in average", then for some large $M > 0$, we should have the inequality

$$\sum_{J\in\mathcal{D},\, J\subset I} |\Delta_J(\varphi w^{\frac12})|^2 \langle v\rangle_J |J| \le M^2 \int_{\mathbb{R}} |\varphi(t)|^2\,dt.$$

Now take $\varphi := w^{\frac12}\chi_I$. Then

$$\varphi w^{\frac12} = w\chi_I, \qquad \int_{\mathbb{R}} |f(t)|^2 dt = \int_I w(t)\,dt,$$

and after division of both parts by $|I|$, the above inequality turns into

$$\sum_{J\in\mathcal{D},\, J\subset I} |\Delta_J w|^2 \langle v\rangle_J \frac{|J|}{|I|} \le M^2 \langle w\rangle_I \qquad (***)$$

What we are going to do now, is to show that for every interval $I\in\mathcal{D}$, for every small $\delta>0$ and for every large $M>0$, one can find a pair of weights $v,w$ satisfying $(*dy)$ and such that $(***)$ fails.

## §3. The Bellman function.

Assume the contrary. Fix some interval $I\in\mathcal{D}$ and two positive numbers $\mathbf{v},\mathbf{w}$ satisfying $\mathbf{v}\cdot\mathbf{w}<1$ (it will be a little bit more technically convenient to work with strong inequalities).

Consider the family of all possible weights $v,w$ satisfying $(*dy)$ and such that $\langle v\rangle_I=\mathbf{v}$, $\langle w\rangle_I=\mathbf{w}$ (the family is non-empty, because, for instance, it contains the pair of identically constant weights $v\equiv\mathbf{v}$ and $w\equiv\mathbf{w}$).

Define

$$\mathcal{B}(\mathbf{v},\mathbf{w}) := \sup \sum_{J\in\mathcal{D}, J\subset I} |\Delta_J w|^2 \langle v\rangle_J \frac{|J|}{|I|},$$

where the supremum is taken over all pairs of weights in our family (since linear change of variable does not modify anything, this supremum does not depend on the choice of the starting interval $I$).

Our assumption means that $\mathcal{B}(\mathbf{v},\mathbf{w})$ is finite for every $\mathbf{v},\mathbf{w}>0$ satisfying $\mathbf{v}\cdot\mathbf{w}<1$, and, moreover, $\mathcal{B}(\mathbf{v},\mathbf{w})\le M^2\mathbf{w}$.

**Properties of the function $\mathcal{B}(\mathbf{v},\mathbf{w})$.**

Domain: $\mathbf{v},\mathbf{w}>0$, $\mathbf{v}\cdot\mathbf{w}<1$.
Range: $0\le\mathcal{B}(\mathbf{v},\mathbf{w})\le M^2\mathbf{w}$.

**The key inequality.**

Let us fix the averages of weights $v$ and $w$ not only over the whole interval $I$, but also over each of the "half-intervals" $I_-$ and $I_+$. Let us denote the corresponding numbers by $\mathbf{v}_\pm$ and $\mathbf{w}_\pm$.

To be consistent, we should assume that $\frac{\mathbf{v}_-+\mathbf{v}_+}{2}=\mathbf{v}$ and $\frac{\mathbf{w}_-+\mathbf{w}_+}{2}=\mathbf{w}$. In order not to break $(*dy)$, we should also demand that

$$\mathbf{v}_\pm\mathbf{w}_\pm<1; \qquad \frac{\mathbf{v}_+}{\mathbf{v}_-}, \frac{\mathbf{w}_+}{\mathbf{w}_-}\in(e^{-\delta},e^{\delta}).$$

The point is (and this is the main advantage of the dyadic model) that in all other respects our choice is free: if we have any two pairs of numbers $(\mathbf{v}_-,\mathbf{w}_-)$ and $(\mathbf{v}_+,\mathbf{w}_+)$, satisfying the above assumptions and if we have two pairs of weights $(v^-,w^-)$ and $(v^+,w^+)$ satisfying $(*dy)$ and such that

$$\langle v^\pm\rangle_{I_\pm}=\mathbf{v}_\pm \quad\text{and}\quad \langle w^\pm\rangle_{I_\pm}=\mathbf{w}_\pm,$$

then the pair of weights

$$v(t) := \begin{cases} \mathbf{v}, & t \notin I; \\ v^-(t), & t \in I_-; \\ v^+(t), & t \in I_+ \end{cases} \qquad \text{and} \qquad w(t) := \begin{cases} \mathbf{w}, & t \notin I; \\ w^-(t), & t \in I_-; \\ w^+(t), & t \in I_+ \end{cases}$$

satisfies $(*dy)$ and $\langle v\rangle_I = \mathbf{v}$, $\langle w\rangle_I = \mathbf{w}$.

Taking this into account and observing that more restrictions mean smaller family of admissible functions and hence not greater supremum, we come to the inequality

$$\mathcal{B}(\mathbf{v},\mathbf{w}) \geq \Big(\frac{\mathbf{w}_+ - \mathbf{w}_-}{2}\Big)^2 \mathbf{v} + \frac{1}{2}\big\{\mathcal{B}(\mathbf{v}_-,\mathbf{w}_-) + \mathcal{B}(\mathbf{v}_+,\mathbf{w}_+)\big\},$$

which we will call the *main inequality.*

The last useful observation is that if a pair of weights $(v, w)$ satisfies $(*dy)$, so does the pair $(\alpha v, \alpha^{-1} w)$, where $\alpha$ is any positive constant. Since

$$\langle \alpha v\rangle_I = \alpha \langle v\rangle_I, \qquad \langle \alpha^{-1} w\rangle_I = \alpha^{-1}\langle w\rangle_I, \quad \text{and} \quad |\Delta_J(\alpha^{-1}w)|^2 \langle \alpha v\rangle_J = \alpha^{-1} |\Delta_J w|^2 \langle v\rangle_J,$$

we get the following "homogeneity" property:

$$\mathcal{B}(\alpha\mathbf{v}, \alpha^{-1}\mathbf{w}) = \alpha^{-1}\mathcal{B}(\mathbf{v},\mathbf{w}).$$

## §4. Regularization of the Bellman function.

We would like to switch to the infinitesimal version of the main inequality, i.e. to the estimate

$$-d^2\mathcal{B} \geq \mathbf{v}(d\mathbf{w})^2.$$

Unfortunately, as defined, the Bellman function $\mathcal{B}$ is clearly measurable, but hardly any better than that (at least, I do not know how to prove that it is differentiable). Of course, one might talk about "generalized" derivatives, but then he would have to explain why it is possible to work with those "generalized" derivatives in the same way as with usual ones. Fortunately, there exists a simple trick that allows to improve an arbitrary Bellman function to a $C^\infty$-smooth "Bellman type function". In our particular situation it works as the following.

Let $\eta : (0, +\infty) \to [0, +\infty)$ be any nonnegative $C^\infty$ function, supported by the interval $[1, 2]$ and such that $\int_0^\infty \eta(t)\,\frac{dt}{t} = 1$. Define

$$\widetilde{\mathcal{B}}(\mathbf{v},\mathbf{w}) := \iint_{(0,+\infty)^2} \mathcal{B}(\tfrac{\mathbf{v}}{s}, \tfrac{\mathbf{w}}{t})\eta(s)\eta(t)\,\frac{ds\,dt}{st} = \iint_{(0,+\infty)^2} \mathcal{B}(s,t)\eta(\tfrac{\mathbf{v}}{s})\eta(\tfrac{\mathbf{w}}{t})\,\frac{ds\,dt}{st}$$

(multiplicative convolution of $\mathcal{B}(\mathbf{v},\mathbf{w})$ with $\eta(\mathbf{v})\eta(\mathbf{w})$).

Note that $\widetilde{\mathcal{B}}$ is well-defined in the region $\{\mathbf{v},\mathbf{w}>0,\ \mathbf{v}\cdot\mathbf{w}<1\}$. The second representation shows that $\widetilde{\mathcal{B}}\in C^\infty$. Meanwhile, the first representation together with the main inequality imply that

$$\widetilde{\mathcal{B}}(\mathbf{v},\mathbf{w})-\frac{1}{2}\{\widetilde{\mathcal{B}}(\mathbf{v}_-,\mathbf{w}_-)+\widetilde{\mathcal{B}}(\mathbf{v}_+,\mathbf{w}_+)\}$$
$$\geq \mathbf{v}\Big(\frac{\mathbf{w}_+-\mathbf{w}_-}{2}\Big)^2\iint_{[1,2]^2}\frac{1}{st^2}\eta(s)\eta(t)\frac{ds\,dt}{st}\geq\frac{1}{8}\mathbf{v}\Big(\frac{\mathbf{w}_+-\mathbf{w}_-}{2}\Big)^2,$$

provided that

$$\frac{\mathbf{v}_-+\mathbf{v}_+}{2}=\mathbf{v},\qquad \frac{\mathbf{w}_-+\mathbf{w}_+}{2}=\mathbf{w};\qquad \frac{\mathbf{v}_+}{\mathbf{v}_-},\frac{\mathbf{w}_+}{\mathbf{w}_-}\in(e^{-\delta},e^{\delta});\qquad \text{and}\quad \mathbf{v}_\pm\mathbf{w}_\pm<1.$$

Note that if $\mathbf{v}\cdot\mathbf{w}<1$, the last two requirements are fulfilled automatically for all $\mathbf{v}_-,\mathbf{v}_+$ sufficiently close to $\mathbf{v}$ and $\mathbf{w}_-,\mathbf{w}_+$ sufficiently close to $\mathbf{w}$. This allows to conclude that

$$-d^2\widetilde{\mathcal{B}}\geq\tfrac{1}{8}\mathbf{v}(d\mathbf{w})^2.$$

Observe also that the double inequality $0\leq\mathcal{B}(\mathbf{v},\mathbf{w})\leq M^2\mathbf{w}$ implies $0\leq\widetilde{\mathcal{B}}(\mathbf{v},\mathbf{w})\leq M^2\mathbf{w}$ and that the homogeneity property of $\mathcal{B}$ is inherited by $\widetilde{\mathcal{B}}$.

## §5. The differential inequality.

The homogeneity property of $\widetilde{\mathcal{B}}$ can be rewritten as

$$\widetilde{\mathcal{B}}(\mathbf{v},\mathbf{w})=\alpha\widetilde{\mathcal{B}}(\alpha v,\alpha^{-1}\mathbf{w}).$$

Putting $\alpha=\mathbf{w}$, we get

$$\widetilde{\mathcal{B}}(\mathbf{v},\mathbf{w})=-\mathbf{w}\xi(\mathbf{v}\cdot\mathbf{w}),$$

where

$$\xi(x):=-\widetilde{\mathcal{B}}(x,1),\qquad 0<x<1.$$

(the minus sign is put just for technical convenience, of course).

Since $\widetilde{\mathcal{B}}$ is infinitely smooth, so is $\xi$. A direct computation yields

$$-d^2\widetilde{\mathcal{B}}=d^2(\mathbf{w}\xi(\mathbf{v}\cdot\mathbf{w}))$$
$$=\mathbf{w}\cdot\begin{pmatrix}\dfrac{d\mathbf{v}}{\mathbf{v}}\\[2mm] \dfrac{d\mathbf{w}}{\mathbf{w}}\end{pmatrix}^T\begin{pmatrix}x^2\xi''(x) & 2x\xi'(x)+x^2\xi''(x)\\[2mm] 2x\xi'(x)+x^2\xi''(x) & 2x\xi'(x)+x^2\xi''(x)\end{pmatrix}\begin{pmatrix}\dfrac{d\mathbf{v}}{\mathbf{v}}\\[2mm] \dfrac{d\mathbf{w}}{\mathbf{w}}\end{pmatrix},$$

where $x=\mathbf{v}\cdot\mathbf{w}$.

Recall that we should have

$$-d^2\widetilde{\mathcal{B}}\geq\tfrac{1}{8}\mathbf{v}(d\mathbf{w})^2.$$

This means that

$$\begin{pmatrix} x^2\xi''(x) & 2x\xi'(x)+x^2\xi''(x) \\ 2x\xi'(x)+x^2\xi''(x) & 2x\xi'(x)+x^2\xi''(x) \end{pmatrix} \geq \begin{pmatrix} 0 & 0 \\ 0 & \frac{x}{8} \end{pmatrix}$$

in the sense of quadratic forms for every $x \in (0,1)$.

In particular, it implies that the quadratic form on the left should be strictly positive definite (the only degenerate forms satisfying this inequality are the forms of the kind $\begin{pmatrix} 0 & 0 \\ 0 & u \end{pmatrix}$ with $u \geq \frac{x}{8}$, but in our quadratic form the out-of-diagonal entries coincide with the entry in the lower right corner, which rules this possibility out).

Strict positive definiteness is equivalent to the conditions

$$x^2\xi''(x) > 0, \qquad 2x\xi'(x)+x^2\xi''(x) > 0, \qquad \text{and} \quad 2x\xi'(x) < 0.$$

From the last inequality we conclude that $\xi'(x) < 0$ for all $x \in (0,1)$. The second one can be written as $(x^2\xi'(x))' > 0$, i.e., the product $x^2\xi'(x)$ should be an increasing function of $x$. Then

$$\xi'(x) \leq \frac{\xi'(\frac{1}{2})}{4x^2}$$

for $0 < x < \frac{1}{2}$. But since $\xi'(\frac{1}{2}) < 0$, this means that $\xi'(x)$ grows not slower than $x^{-2}$ as $x$ approaches 0, which is a little bit faster than a derivative of a bounded function can afford. This contradiction finishes the proof.

## §6. From dyadic $\delta$-smoothness to full $\delta$-smoothness.

As we have already mentioned above, the dyadic version of the $\delta$-smoothness condition does not imply the full version of it. Indeed, if $\mathcal{D}$ is the standard dyadic lattice, consisting of intervals $[\frac{k}{2^n}, \frac{k+1}{2^n}]$, $k, n \in \mathbb{Z}$, then every weight $v$ constant on both semiaxes $(-\infty, 0]$ and $[0, +\infty)$ is 0-smooth in the sense of the dyadic condition, but the ratio $\frac{\langle v\rangle_{[0,1]}}{\langle v\rangle_{[-1,0]}}$ can be as large (or as small) as one wants! Still the hierarchy of dyadic intervals is a so pleasant feature that to sacrifice it completely would be a pity. So, let us show that we can check $\delta$-smoothness of weights using dyadic intervals only.

**Lemma:**

Let $\delta > 0$ be a very small number. Let $v$ be a weight. Assume that $\frac{\langle v\rangle_{I_1}}{\langle v\rangle_{I_2}} \in (e^{-\delta}, e^{\delta})$ for every two adjacent dyadic intervals $I_1, I_2 \in \mathcal{D}$ of equal length. (From now on, we will call this property *strong* dyadic $\delta$-smoothness; note that the union $I_1 \cup I_2$ may be not a dyadic interval now!).

Then $v$ is $10\sqrt{\delta}$-smooth.

**Proof:**

Let $I \subset \mathbb{R}$ be any interval. Let $J \in \mathcal{D}$ be a dyadic interval such that $J \cap I \neq \varnothing$ and $\frac{\sqrt{\delta}}{2}|I| \le |J| \le \sqrt{\delta}|I|$. Our aim will be to show that

$$\frac{\langle v\rangle_J}{\langle v\rangle_I} \in (e^{-5\sqrt{\delta}}, e^{5\sqrt{\delta}}).$$

Let $\mathcal{J}_*$ be the family of all dyadic intervals of the same length as $J$ that are contained in $I$. The number of intervals in $\mathcal{J}_*$ does not exceed $\frac{|I|}{|J|} \le \frac{2}{\sqrt{\delta}}$. Since $J \cap I \neq \varnothing$, we have

$$\langle v\rangle_{J'} \ge e^{-\frac{2}{\sqrt{\delta}}\delta}\langle v\rangle_J = e^{-2\sqrt{\delta}}\langle v\rangle_J.$$

Let $I_* = \bigcup \mathcal{J}_*$. Clearly, $I_*$ is an interval and $|I_*| \ge |I| - 2|J| \ge (1-2\sqrt{\delta})|I| \ge e^{-3\sqrt{\delta}}|I|$, provided that $\delta$ is small enough. Hence, we can write

$$\langle w\rangle_I = \frac{1}{|I|}\int_I v \ge \frac{1}{|I|}\int_{I_*} v = \frac{|J|}{|I|}\sum_{J' \in \mathcal{J}_*}\langle v\rangle_{J'} \ge \frac{|J|}{|I|}\frac{|I_*|}{|J|}e^{-2\sqrt{\delta}}\langle v\rangle_J \ge e^{-5\sqrt{\delta}}\langle v\rangle_J.$$

To get the estimate from above, consider the family of all dyadic intervals of the same length as $J$ intersecting $I$ and their union $I^*$. The number of such intervals equals $\frac{|I^*|}{|J|} \le \frac{|I|+2|J|}{|J|} \le \frac{2}{\sqrt{\delta}} + 2$. Since $J$ is one of them, we get

$$\langle v\rangle_{J'} \le e^{(\frac{2}{\sqrt{\delta}}+1)\delta}\langle v\rangle_J = e^{2\sqrt{\delta}+\delta}\langle v\rangle_J \le e^{3\sqrt{\delta}}\langle v\rangle_J.$$

Hence,

$$\langle v\rangle_I = \frac{1}{|I|}\int_I v \le \int_{I_*} v = \frac{|J|}{|I|}\sum_{J' \in \mathcal{J}^*}\langle v\rangle_{J'} \ge \frac{|J|}{|I|}\frac{|I^*|}{|J|}e^{3\sqrt{\delta}}\langle v\rangle_J \ge e^{5\sqrt{\delta}}\langle v\rangle_J,$$

because $|I^*| \le |I| + 2|J| \le (1+2\sqrt{\delta})|I| \le e^{2\sqrt{\delta}}|I|$.

Now, let $I \subset \mathbb{R}$ be any interval. Let $J$ be the dyadic interval of length between $\frac{\sqrt{\delta}}{4}|I|$ and $\frac{\sqrt{\delta}}{2}|I|$ containing the center of $I$. Then

$$\frac{\langle v\rangle_{I_\pm}}{\langle v\rangle_J} \in (e^{-5\sqrt{\delta}}, e^{5\sqrt{\delta}}),$$

and therefore

$$\frac{\langle v\rangle_{I_+}}{\langle v\rangle_{I_-}} \in (e^{-10\sqrt{\delta}}, e^{10\sqrt{\delta}}),$$

i.e., $v$ is $10\sqrt{\delta}$-smooth.

## §7. The main step of the construction.

What can we gain by considering dyadic intervals only? The point is that now we can change our weights on dyadic intervals without spoiling their smoothness. Namely, we can do the following operation.

Assume that $v$ is a weight, $I$ is a dyadic interval and that $v \equiv \mathbf{v}$ on $I$. Choose any two positive numbers $\mathbf{v}_-$ and $\mathbf{v}_+$ such that $\frac{\mathbf{v}_- + \mathbf{v}_+}{2} = \mathbf{v}$ and $\frac{\mathbf{v}_+}{\mathbf{v}_-} \in (e^{-\delta}, e^{\delta})$. Take any large integer $N$ and consider the dyadic intervals $J \subset I$ of length $2^{-N}|I|$. Now for every such an interval $J$ that lies "deep inside" $I$ (i.e., does not contain an endpoint of $I$), change the weight $v$ (which originally is equal to $\mathbf{v}$ on the whole interval $J$) to $\mathbf{v}_-$ on $J_-$ and to $\mathbf{v}_+$ on $J_+$ (See Figure 1, where the case $N = 3$, $\mathbf{v}_- < \mathbf{v}_+$ is shown).

If the original weight $v$ satisfied the strong dyadic $\delta$-smoothness condition, then the same is true for the new weight. Indeed, the average of $v$ over any dyadic interval of length $2^{-N}|I|$ or more has remained unchanged. On the other hand, if $I_1$ and $I_2$ are two adjacent dyadic intervals of equal length $|I_1| = |I_2| < 2^{-N}|I|$, then the average over any of the intervals $I_1$ and $I_2$ might change only if *both* of them are contained in $I$. But then both the corresponding averages are between $\mathbf{v}_-$ and $\mathbf{v}_+$ and therefore their ratio is between $e^{-\delta}$ and $e^{\delta}$ again!

## §8. Weights as martingales.

Let $v$ be a weight. Suppose that we are interested in its behavior on some dyadic interval $I$. We may start with the constant weight $v_0 \equiv \langle v \rangle_I$ as a rough approximation to $v$. Then we may improve it to the weight $v_1$, which equals $\langle v \rangle_{I_-}$ on $I_-$ and $\langle v \rangle_{I_+}$ on $I_+$. The next step would be to consider the left and the right halves $I_{--}$ and $I_{-+}$ of the interval $I_-$ and to improve the weight $v_1$ to the weight $v_2$, which is identically equal to $\langle v \rangle_{I_{--}}$ on $I_{--}$, to $\langle v \rangle_{I_{-+}}$ on $I_{-+}$, and to $\langle v \rangle_{I_+}$ on $I_+$. Then we may split the interval $I_+$ into $I_{+-}$ and $I_{++}$ and obtain the weight $v_3$, and so on.

What we do at the $k$-th step is the following. We take the weight $v_{k-1}$ and some dyadic interval $J$, on which $v_{k-1}$ is constant ($I$ at the first step, $I_-$ at the second, $I_+$ at the third, $I_{--}$ at the fourth, etc.), and then construct the weight $v_k$ by changing $v_{k-1} \equiv \langle v \rangle_J$ on $J$ to $\langle v \rangle_{J_-}$ on $J_-$ and $\langle v \rangle_{J_+}$ on $J_+$. Every next weight $v_k$ is a better approximation to the weight $v$ than the previous weight $v_{k-1}$. Actually, it is possible to show that the weights $v_k$ converge to $v$ in any reasonable sense.

Of course, the weights $v_k$ on can also be considered as the partial sums of the series

$$\langle v \rangle_I + \sum_{J \in \mathcal{D}\,,\, J \subset I} \Delta_J v \, h_I,$$

where the order of summation is "natural": $J'$ precedes $J''$ if either $|J'| > |J''|$, or $|J'| = |J''|$ and $J'$ lies to the left from $J''$.

Note that the sequence of weights $\{v_k\}$ is completely determined by the averages $\langle v \rangle_J$ : $J \in \mathcal{D}, J \subset I$. In the next section we are going to use the same family of numbers $\langle v \rangle_J$ to construct *another* sequence of weights $\{\widetilde{v}_k\}$. The main difference is that in order to get $\widetilde{v}_k$ from $\widetilde{v}_{k-1}$, we will now apply the procedure described in §7.

## §9. Remodeling.

Before starting the formal (a little bit more formal than desirable) construction of the weights $\widetilde{v}_k$, we would like to attract the attention of the reader to Figure 2, where the weights $\widetilde{v}_0$, $\widetilde{v}_1$, $\widetilde{v}_2$, and $\widetilde{v}_3$ are shown versus the weights $v_0$, $v_1$, $v_2$ and $v_3$. It is very likely that the reader will get the whole idea of remodeling from this picture and the rest of this section will serve him just as a "notation list". And of course, it was this simple picture and not the awkward English text below, which appeared before our eyes when we understood how to switch from the Haar multipliers to the Hilbert transform for the first time.

Let $v$ be a weight and let $I \in \mathcal{D}$ be a dyadic interval. Let also $\widetilde{I}$ be some other (maybe, the same) dyadic interval. The construction of weights $\widetilde{v}_k$ goes by induction as the following.

For the $k$-th step we will have the following three objects:

**1)** a dyadic interval $J \subset I$ (which will be just the $k$-th interval in the "natural" ordering $I,\ I_-, I_+,\ I_{--}, I_{-+}, I_{+-}, \dots$);

**2)** a finite family $\mathcal{F}(J)$ of pairwise disjoint dyadic subintervals of $\widetilde{I}$, "associated" with $J$;

**3)** the weight $\widetilde{v}_{k-1}$, which is identically equal to $\langle v\rangle_J$ on every interval in $\mathcal{F}(J)$.

To start the construction, we put $\widetilde{v}_0 \equiv \langle v\rangle_I$ (on the whole real line), $J = I$ and $\mathcal{F}(J) = \{\widetilde{I}\}$.

To get $\widetilde{v}_k$, we will choose some large positive integer $N_k$ (it will be discussed below, where exactly this choice comes from) and apply the "main step" from §7 on *every* interval $\widetilde{J} \in \mathcal{F}(J)$ with $N = N_k$, $\mathbf{v}_- = \langle v\rangle_{J_-}$ and $\mathbf{v}_+ = \langle v\rangle_{J_+}$.

Now, for every interval $\widetilde{J} \in \mathcal{F}(J)$, the weight $\widetilde{v}_k$ identically equals $\langle v\rangle_{J_-}$ on the left half of each dyadic subinterval of $\widetilde{J}$ of length $2^{-N_k}|\widetilde{J}$ that does not contain an endpoint of $\widetilde{J}$. All these left halves (coming from *all* the intervals $\widetilde{J} \in \mathcal{F}(J)$ ) will constitute the family $\mathcal{F}(J_-)$. The family $\mathcal{F}(J_+)$ will, naturally, consist of the corresponding right halves.

Of course, in most cases none of the families $\mathcal{F}(J_-)$ and $\mathcal{F}(J_+)$ will be used right during the next ($k+1$-st) step. Most likely, the intervals $J_-$ and $J_+$ will be considered during the steps $\ell$ and $\ell+1$ respectively with $\ell$ much greater than $k$. We leave it to the reader as an exercise to show that our inductive construction is, nevertheless, consistent in the sense that the weight $\widetilde{v}_{\ell-1}$ will still be identically equal to $\langle v\rangle_{J_-}$ on each interval in the family $\mathcal{F}(J_-)$, etc. The reason is that the families $\mathcal{F}(J)$ inherit all the hierarchy of the dyadic intervals $J \subset I$. Namely, if we denote by $E(J)$ the union of all intervals in the family $\mathcal{F}(J)$, we shall have $J' \subset J'' \Longrightarrow E(J') \subset E(J'')$ and $J' \cap J'' = \varnothing \Longrightarrow E(J') \cap E(J'') = \varnothing$.

Also, let us observe (though it will be of no *formal* importance) that, as constructed, every family $\mathcal{F}(J)$ consists of dyadic intervals of the same length (different for different $J$, of course).

## §10. Properties of the weights $\widetilde{v}_k$.

**A)** The remodeling improves the smoothness of the weight $v$. Namely, if the original weight $v$ satisfied the dyadic $\delta$-smoothness condition, then, as we saw in §7, for each $k$ the weight $\widetilde{v}_k$ satisfies the strong dyadic $\delta$-smoothness condition and thereby, according to the result of §6, is $10\sqrt{\delta}$-smooth.

**B)** The weights $\widetilde{v}_k$ have the same averages over dyadic intervals as the original weight $v$. More precisely, for any dyadic interval $\widetilde{J} \in \mathcal{D}$ and any $k \geq 0$, there exists an interval $J \in \mathcal{D}$ such that $\langle \widetilde{v}_k \rangle_{\widetilde{J}} = \langle v \rangle_J$.

The reader can easily convince himself in that just looking at Figure 2 and experimenting with various dyadic intervals. In few seconds he will see what happens. As to the formal proof, it can be done by induction as the following. We have $\langle \widetilde{v}_0 \rangle_{\widetilde{J}} = \langle v \rangle_I$ for every $\widetilde{J}$ (just because $\widetilde{v}_0 \equiv \langle v \rangle_I$ on the whole real line). Now, the average $\langle \widetilde{v}_k \rangle_{\widetilde{J}}$ can differ from the average $\langle \widetilde{v}_{k-1} \rangle_{\widetilde{J}}$ only if $\widetilde{J}$ is a subinterval of one of the intervals in $\mathcal{F}(J_-)$ or $\mathcal{F}(J_+)$, where $J \subset I$ is the dyadic interval we deal with during the $k$-th step. Then $\langle \widetilde{v}_k \rangle_{\widetilde{J}} = \langle v \rangle_{J_-}$ in the first case, and $\langle \widetilde{v}_k \rangle_{\widetilde{J}} = \langle v \rangle_{J_+}$ in the second one.

For a dyadic interval $\widetilde{J} \subset \widetilde{I}$, the corresponding interval $J$ can also be defined as the shortest interval among the first $k$ intervals in the sequence $I, I_-, I_+, I_{--}, I_{-+}, I_{+-}, \ldots$ that satisfies $\widetilde{J} \subset E(J)$ (for any $\widetilde{J} \not\subset \widetilde{I}$, one can take $J = I$).

Anyhow, the point is that such an interval $J$ always exists and, what is more, depends on the interval $\widetilde{J}$, number $k$, the starting intervals $I$ and $\widetilde{I}$ and the sequence of numbers $N_k$ used in the construction, but *not on the weight* $v$. In particular, if we take a pair of weights $v$ and $w$, satisfying the dyadic Muckenhoupt condition and consider their remodelings $\widetilde{v}_k$ and $\widetilde{w}_k$, corresponding to the same starting intervals $I$ and $\widetilde{I}$ and the same sequence of numbers $N_k$, then for each $k$, the pair of weights $\widetilde{v}_k$ and $\widetilde{w}_k$ will also satisfy the dyadic Muckenhoupt condition with the same constant as $v$ and $w$.

## §11. From Haar multipliers to the Hilbert transform.

Now take the pair of weights $v$ and $w$, "constructed" in §§2-5 and consider their remodelings $\widetilde{v}_k$ and $\widetilde{w}_k$ (we take the same $I$, $\widetilde{I}$, and $N_k$ for both $v$ and $w$).

Since the weights $v$ and $w$ satisfied the dyadic $\delta$-smoothness condition, we conclude that all the weights $\widetilde{v}_k$ and $\widetilde{w}_k$ satisfy the strong dyadic $\delta$-smoothness condition and therefore are $10\sqrt{\delta}$-smooth.

Since $\langle v \rangle_J \langle w \rangle_J \leq 1$ for every $J \in \mathcal{D}$, we also have $\langle \widetilde{v}_k \rangle_J \langle \widetilde{w}_k \rangle_J \leq 1$ for every $J \in \mathcal{D}$ and for each $k \geq 0$. Now take any (not necessarily dyadic) interval $J' \subset \mathbb{R}$. Pick any dyadic interval $J \in \mathcal{D}$ of length $\frac{\sqrt{\delta}}{2}|J'| \leq |J| \leq \sqrt{\delta}|J'|$ such that $J \cap J' \neq \varnothing$. Since the weights $\widetilde{v}_k$ and $\widetilde{w}_k$ satisfy the strong dyadic $\delta$-smoothness condition, we obtain (see §6 for details) that

$$\langle \widetilde{v}_k \rangle_{J'} \leq e^{5\sqrt{\delta}} \langle \widetilde{v}_k \rangle_J \qquad \text{and} \qquad \langle \widetilde{w}_k \rangle_{J'} \leq e^{5\sqrt{\delta}} \langle \widetilde{w}_k \rangle_J .$$

Thus

$$\langle \widetilde{v}_k \rangle_{J'} \langle \widetilde{w}_k \rangle_{J'} \leq e^{10\sqrt{\delta}} \langle \widetilde{v}_k \rangle_J \langle \widetilde{w}_k \rangle_J \leq e^{10\sqrt{\delta}},$$

i.e., the pair of weights $\widetilde{v}_k$ and $\widetilde{w}_k$ satisfies the full (not just dyadic) Muckenhoupt condition with the constant $e^{10\sqrt{\delta}} < 2$ (provided that $\delta > 0$ is small enough).

We are going to show now that the norm of the weighted Hilbert transform $\widetilde{H} = \widetilde{v}_\ell^{\frac{1}{2}} H \widetilde{w}_\ell^{\frac{1}{2}}$ is at least $\frac{M}{2}$ for all large $\ell$. To this end, consider the test function $\psi = \widetilde{w}_\ell^{\frac{1}{2}} \chi_{\widetilde{I}}$.

We have

$$\|\psi\|^2_{L^2(\mathbb{R})} = \int_{\widetilde{I}} \widetilde{w}_\ell(t)dt = |\widetilde{I}|\langle \widetilde{w}_\ell \rangle_{\widetilde{I}} = |\widetilde{I}|\langle w \rangle_I$$

regardless of $\ell$.

On the other hand,

$$\|\widetilde{H}\psi\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} |\widetilde{H}\psi|^2 = \int_{\mathbb{R}} |H(\widetilde{w}_\ell \chi_{\widetilde{I}})|^2 \, \widetilde{v}_\ell$$

$$\geq \int_{\mathbb{R}} |H(\widetilde{w}_\ell \chi_{\widetilde{I}})|^2 \, \widetilde{v}_\ell - \int_{\mathbb{R}} |H(\widetilde{w}_0 \chi_{\widetilde{I}})|^2 \, \widetilde{v}_0 = \sum_{k=1}^{\ell} \Big[ \int_{\mathbb{R}} |H(\widetilde{w}_k \chi_{\widetilde{I}})|^2 \, \widetilde{v}_k - \int_{\mathbb{R}} |H(\widetilde{w}_{k-1} \chi_{\widetilde{I}})|^2 \, \widetilde{v}_{k-1} \Big].$$

Pick some sequence of positive numbers $\varepsilon_k$, such that

$$\sum_{k=1}^{\infty} \varepsilon_k \leq \frac{M^2}{4} \langle w \rangle_I |\widetilde{I}|.$$

Our aim will be to show that if the numbers $N_k$ increase fast enough, then for each $k \geq 1$, the difference

$$\int_{\mathbb{R}} |H(\widetilde{w}_k \chi_{\widetilde{I}})|^2 \, \widetilde{v}_k - \int_{\mathbb{R}} |H(\widetilde{w}_{k-1} \chi_{\widetilde{I}})|^2 \, \widetilde{v}_{k-1} \geq \frac{1}{2} |\Delta_J w|^2 \langle v \rangle_J \frac{|J|}{|I|} |\widetilde{I}| - \varepsilon_k,$$

where $J \subset I$ is the dyadic interval we deal with during the $k$-th step.

Since in our remodeling procedure we go one by one over all dyadic subintervals $J$ of the interval $I$ and since

$$\sum_{J \in \mathcal{D},\, J \subset I} |\Delta_J w|^2 \langle v \rangle_J \tfrac{|J|}{|I|} > M^2 \langle w \rangle_I,$$

this would imply that

$$\|\widetilde{H}\psi\|^2_{L^2(\mathbb{R})} > \frac{M^2}{2} \langle w \rangle_I |\widetilde{I}| - \sum_{k=1}^{\infty} \varepsilon_k \geq \frac{M^2}{2} \langle w \rangle_I |\widetilde{I}| - \frac{M^2}{4} \langle w \rangle_I |\widetilde{I}| = \frac{M^2}{4} \|\psi\|^2_{L^2(\mathbb{R})}$$

for sufficiently large $\ell$, finishing the story.

## §12. The estimate of the increment $\int_{\mathbb{R}} |H(\widetilde{w}_k \chi_{\widetilde{I}})|^2 \, \widetilde{v}_k - \int_{\mathbb{R}} |H(\widetilde{w}_{k-1} \chi_{\widetilde{I}})|^2 \, \widetilde{v}_{k-1}$.

We will start with a couple of preliminary observations.

First of all, notice that the weight $\widetilde{v}_k$ is nowhere essentially less than the weight $\widetilde{v}_{k-1}$. Namely, for each $k$,

$$\widetilde{v}_k(t) \geq e^{-\delta} \widetilde{v}_{k-1}(t) \qquad \text{for every } t \in \mathbb{R}.$$

Indeed, $\widetilde{v}_k(t)$ differs from $\widetilde{v}_{k-1}(t)$ only if $t \in E(J)$ (where $J \subset I$ is the dyadic interval we deal with during the $k$-th step). But then both the values $\widetilde{v}_k(t)$ and $\widetilde{v}_{k-1}(t)$ are between $\langle v \rangle_{J_-}$ and $\langle v \rangle_{J_+}$, and therefore their ratio is between $e^{-\delta}$ and $e^{\delta}$.

Secondly, let us observe that for any dyadic interval $J \in \mathcal{D}$

$$|E(J)| \geq \prod_{j=1}^{\infty}(1 - 2 \cdot 2^{-N_j}) \times \frac{|\widetilde{I}|}{|I|}|J|.$$

Indeed, if an interval $J \subset I$ is considered during the $k$-th step, then

$$|E(J_-)| = |E(J_+)| = \tfrac{1}{2}(1 - 2 \cdot 2^{-N_k})|E(J)|.$$

From here and from the identity $|E(I)| = |\widetilde{I}|$, one can easily derive that for any dyadic interval $J \subset I$, the ratio $\frac{|E(J)|}{|J|}$ is equal to $\frac{|\widetilde{I}|}{|I|}$ times the product of several different factors in the infinite product $\prod_{j=1}^{\infty}(1 - 2 \cdot 2^{-N_j})$ (namely, the factors corresponding to the steps during which the dyadic intervals $J'$ satisfying $J \subset J' \subset I$, $J' \neq J$ were considered)

If we demand that $N_j \geq j + 3$ for every $j \geq 1$, then we will have

$$\prod_{j=1}^{\infty}(1 - 2 \cdot 2^{-N_j}) \geq \prod_{j=1}^{\infty}(1 - 2^{-(j+2)}) \geq 1 - \sum_{j=1}^{\infty} 2^{-(j+2)} = \frac{3}{4}$$

and thereby

$$|E(J)| \geq \frac{3}{4}\frac{|\widetilde{I}|}{|I|}|J|$$

for any dyadic interval $J \subset I$.

Let now fix $k \geq 1$ and let $J \subset I$ be the dyadic interval we deal with at the $k$-th step. Observe that

$$\widetilde{v}_k = \widetilde{v}_{k-1} + (\Delta_J v)\varphi_J \qquad \text{and} \qquad \widetilde{w}_k = \widetilde{w}_{k-1} + (\Delta_J w)\varphi_J,$$

where

$$\varphi_J(t) = \begin{cases} -1, & \text{if } t \in E(J_-); \\ +1, & \text{if } t \in E(J_+); \\ 0, & \text{otherwise} \end{cases}$$

(see Figure 3).

Since $\operatorname{supp} \varphi_J \subset E(J) \subset \widetilde{I}$, we get

$$\int_{\mathbb{R}} |H(\widetilde{w}_k \chi_{\widetilde{I}})|^2 \, \widetilde{v}_k = \int_{\mathbb{R}} |H[\widetilde{w}_{k-1}\chi_{\widetilde{I}} + (\Delta_J w) H\varphi_J]|^2 \, \widetilde{v}_k$$

$$= \int_{\mathbb{R}} |H(\widetilde{w}_{k-1}\chi_{\widetilde{I}})|^2 \, \widetilde{v}_k + 2(\Delta_J w) \int_{\mathbb{R}} H(\widetilde{w}_{k-1}\chi_{\widetilde{I}}) \cdot H\varphi_J \cdot \widetilde{v}_k + |\Delta_J w|^2 \int_{\mathbb{R}} |H\varphi_J|^2 \, \widetilde{v}_k$$

$$= \int_{\mathbb{R}} |H(\widetilde{w}_{k-1}\chi_{\widetilde{I}})|^2 \, \widetilde{v}_{k-1} + (\Delta_J v) \int_{\mathbb{R}} |H(\widetilde{w}_{k-1}\chi_{\widetilde{I}})|^2 \, \varphi_J$$
$$+ 2(\Delta_J w) \int_{\mathbb{R}} H(\widetilde{w}_{k-1}\chi_{\widetilde{I}}) \cdot H\varphi_J \cdot \widetilde{v}_{k-1} + 2(\Delta_J w)(\Delta_J v) \int_{\mathbb{R}} H(\widetilde{w}_{k-1}\chi_{\widetilde{I}}) \cdot H\varphi_J \cdot \varphi_J$$
$$+ |\Delta_J w|^2 \int_{\mathbb{R}} |H\varphi_J|^2 \, \widetilde{v}_k$$

$$=: \int_{\mathbb{R}} |H(\widetilde{w}_{k-1}\chi_{\widetilde{I}})|^2 \, \widetilde{v}_{k-1} + (\Delta_J v)\mathcal{I}_1 + 2(\Delta_J w)\mathcal{I}_2 + 2(\Delta_J w)(\Delta_J v)\mathcal{I}_3 + |\Delta_J w|^2 \mathcal{I}_4.$$

The only freedom we have during the $k$-th step of our construction is the choice of the number $N_k$. The greater this number is, the faster the function $\varphi_J$ oscillates. Let $\varphi_{J,n}$ be the function, corresponding to the choice $N_k = n$. It is clear that the functions $\varphi_{J,n}$ weakly converge to 0 over any $L^p(\mathbb{R})$ with $1 \le p < +\infty$ as $n \to \infty$.

This allows to immediately obtain that the integral $\mathcal{I}_1$ tends to 0 as $n \to \infty$ (the factor $|H(\widetilde{w}_{k-1}\chi_{\widetilde{I}})|^2$ is a *fixed* (i.e., not depending on $n$) $L^1$-function).

Also, since $H$ is a bounded operator in $L^2(\mathbb{R})$, we conclude that the functions $H\varphi_{J,n}$ weakly tend to 0 over $L^2(\mathbb{R})$, and therefore $\mathcal{J}_2 \to 0$ as $n \to \infty$ (now $H(\widetilde{w}_{k-1}\chi_{\widetilde{I}}) \in L^2(\mathbb{R})$ and $\widetilde{v}_{k-1} \in L^\infty(\mathbb{R})$, so their product is a fixed $L^2$-function).

Thus, to get the desided estimate, we only need to take care of $\mathcal{I}_3$ and $\mathcal{I}_4$ for large $n$.

## §13. Estimate of $\mathcal{I}_4$.

We have

$$\mathcal{I}_4 = \int_{\mathbb{R}} |H\varphi_{J,n}|^2 \, \widetilde{v}_k \ge e^{-\delta} \int_{\mathbb{R}} |H\varphi_{J,n}|^2 \, \widetilde{v}_{k-1}$$
$$\ge e^{-\delta} \int_{E(J)} |H\varphi_{J,n}|^2 \, \widetilde{v}_{k-1} = e^{-\delta} \langle v \rangle_J \int_{E(J)} |H\varphi_{J,n}|^2$$

(recall that $\widetilde{v}_{k-1} \equiv \langle v \rangle_J$ on $E(J)$ ).

Since $H$ is a unitary operator, we obtain

$$\int_{\mathbb{R}} |H\varphi_{J,n}|^2 = \int_{\mathbb{R}} |\varphi_{J,n}|^2 \to |E(J)| \qquad \text{as } n \to \infty.$$

On the other hand, for every point $t \notin E(J)$,

$$|H\varphi_{J,n}(t)| = \Big|\frac{1}{\pi}\int_{\mathbb{R}} \frac{\varphi_{J,n}(s)\,ds}{t-s}\Big| \le \frac{1}{\pi}\int_{E(J)} \frac{ds}{|t-s|} = \frac{1}{\pi}\sum_{\widetilde{J}\in\mathcal{F}(J)} \int_{\widetilde{J}} \frac{ds}{|t-s|} = \sum_{\widetilde{J}\in\mathcal{F}(J)} |H\chi_{\widetilde{J}}(t)|$$

and

$$H\varphi_{J,n}(t) \to 0 \qquad \text{as } n \to \infty.$$

Since $\mathcal{F}(J)$ is a *finite* family of intervals and since $H\chi_{\widetilde{J}} \in L^2(\mathbb{R})$ for any interval $\widetilde{J} \in \mathcal{F}(J)$, we can use the dominated convergence theorem to conclude that

$$\int_{\mathbb{R}\setminus E(J)} |H\varphi_{J,n}|^2 \to 0 \qquad \text{as } n \to \infty.$$

Therefore

$$\int_{E(J)} |H\varphi_{J,n}|^2 \to |E(J)| \qquad \text{as } n \to \infty,$$

and the integral $\mathcal{I}_4$ can be estimated (up to "epsilon") from below by

$$e^{-\delta}\langle v\rangle_J |E(J)| \ge e^{-\delta}\langle v\rangle_J \tfrac{3}{4}\tfrac{|\widetilde{I}|}{|I|}|J| \ge \tfrac{1}{2}\langle v\rangle_J \tfrac{|\widetilde{I}|}{|I|}|J|,$$

provided that $\delta > 0$ is small enough. This estimate, together with the factor $|\Delta_J w|^2$ in front of $\mathcal{I}_4$, gives us the term $\frac{1}{2}|\Delta_J w|^2\langle v\rangle_J \frac{|\widetilde{I}|}{|I|}|J|$ desired.

## §14. Estimate of $\mathcal{I}_3$

Our aim will be to show that $\mathcal{I}_3 \to 0$ as $n \to \infty$. It is enough to prove that the products $\varphi_{J,n} \cdot H\varphi_{J,n}$ weakly tend to 0 over $L^2(\mathbb{R})$ as $n \to \infty$.

Since $\|\varphi_{J,n}\|_{L^\infty(\mathbb{R})} \le 1$ and $\|H\varphi_{J,n}\|_{L^2(\mathbb{R})} = \|\varphi_{J,n}\|_{L^2(\mathbb{R})} \le |E(J)|$, the $L^2$-norms of these products stay uniformly bounded. So, it remains only to check that for each interval $Q \subset \mathbb{R}$,

$$\lim_{n\to\infty} \int_Q \varphi_{J,n} \cdot H\varphi_{J,n} = 0.$$

Split the functions $\varphi_{J,n}$ as

$$\varphi_{J,n} = \varphi_{J,n}\chi_Q + \varphi_{J,n}\chi_{\mathbb{R}\setminus Q} =: \varphi'_{J,n} + \varphi''_{J,n}.$$

Obviously, both $\varphi'_{J,n}$ and $\varphi''_{J,n}$ weakly tend to 0 over any $L^p(\mathbb{R})$ $(1 \le p < \infty)$.

We have

$$\int_Q \varphi_{J,n} \cdot H\varphi_{J,n} = \int_Q \varphi'_{J,n} \cdot H\varphi'_{J,n} + \int_Q \varphi'_{J,n} \cdot H\varphi''_{J,n}$$

(the products containing the factor $\varphi''_{J,n}$ vanish on $Q$).

The same reasoning as in the previous section shows that $\lim_{n\to\infty}\int_Q |H\varphi''_{J,n}|^2 = 0$ and thereby the second integral is as close to 0 as one desires for large $n$ (the integrals $\int_Q |\varphi'_{J,n}|^2$ are all bounded by $|E(J)|$ or $|Q|$, whichever the reader prefers).

As to the first integral, here a little miracle happens: since $H$ is *antisymmetric*,

$$\int_Q \varphi'_{J,n} \cdot H\varphi'_{J,n} = \int_{\mathbb{R}} \varphi'_{J,n} \cdot H\varphi'_{J,n} = 0$$

regardless of $n$!

**§15. One more remark.**