MTH 370, Fall 2009
Solutions to Midterm

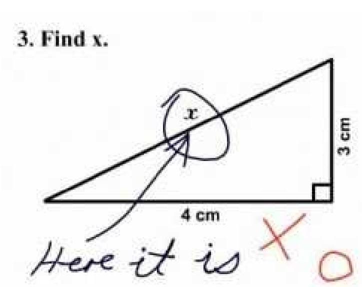
Name:

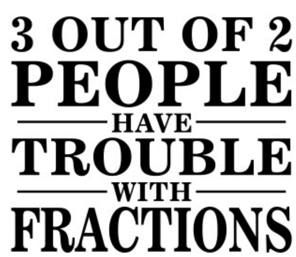
Instructions:

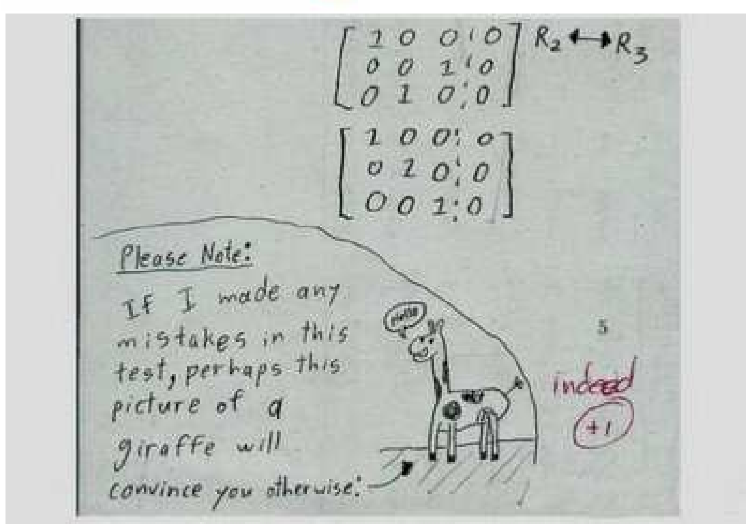
1. Print your name in the space provided above.
2. There are two problems. Do these problems by hand (no calculators, computers, etc.) and show your work on the pages provided (no additional scratch paper).
3. You may begin the exam when Berton indicates it is 12:40pm.
4. All exams must be returned by the end of the regular class period (1:30 pm).
5. Try to enjoy yourself.

Here are some useful reminders about taking derivatives:

Rule	Formula	Example
Product	$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$	$(x\ln(x))' = x'\ln(x) + x\ln'(x) = \ln(x) + 1$
Quotient	$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$	$\left(\frac{x}{x+1}\right)' = \frac{x'(x+1)-x(x+1)'}{(x+1)^2} = \frac{1}{(x+1)^2}$
Chain	$(f(g(x)))' = f'(g(x))g'(x)$	$(e^{-x^2})' = e^{-x^2}(-x^2)' = -2xe^{x^2}$







1. Consider the Beverton-Holt model,

$$x_{n+1} = \frac{rx_n}{x_n + 1} \quad (r > 1), \tag{1}$$

which has been used to successfully model some fish populations.

   (a) (20%) Find the two fixed points of (1).

   (b) (20%) Determine the stability of the fixed points.

Solutions:

   (a) x is a fixed point of (1) if and only if

$$x = \frac{rx}{x+1}.$$

   Therefore, either $x = 0$ or we can divide the above equation by x to get

$$\begin{aligned} 1 &= \frac{r}{x+1} \\ x + 1 &= r \\ x &= r - 1 \end{aligned}$$

   Since $r > 1$, the fixed point $x = r - 1 > 0$.

   (b) The right-hand side of (1) is

$$f(x) = \frac{rx}{x+1}.$$

   The derivative of f is

$$f'(x) = \frac{r}{(x+1)^2},$$

   thus since $r > 1$,

$$|f'(0)| = r > 1, \quad |f'(r-1)| = \frac{r}{r^2} = \frac{1}{r} < 1.$$

   Therefore $x = 0$ is unstable and $x = r - 1$ is stable.

2. Consider the following age-structured population model,

$$\mathbf{x}_{n+1} = M\mathbf{x}_n \quad \left(M = \begin{bmatrix} 1 & 0.5 \\ 1 & 0.5 \end{bmatrix}, \quad \mathbf{x}_n = \begin{bmatrix} A_n \\ J_n \end{bmatrix} \right). \tag{2}$$

(a) (20%) Compute the trace, determinant and inverse of M.

(b) (30%) Compute the eigenvalues and eigenvectors of M.

(c) (10%) Will this population eventually go extinct?

Solutions:

(a) $\text{tr}(M) = 1 + 0.5 = 1.5$, $\det(M) = 1 \cdot 0.5 - 1 \cdot 0.5 = 0$, and M has no inverse since $\det(M) = 0$.

(b) The characteristic equation is

$$0 = \det(M - \lambda I) = \lambda^2 - \text{tr}(M)\lambda + \det(M) = \lambda(\lambda - \text{tr}(M)),$$

therefore the eigenvalues of M are $\lambda_+ = 1.5$ and $\lambda_- = 0$.
The eigenvector $\mathbf{x}_+$ associated with $\lambda_+ = 1.5$ satisfies

$$\mathbf{0} = (M - \lambda_+ I)\mathbf{x}_+ = \begin{bmatrix} -0.5 & 0.5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Each of the equations represented above implies $x_2 = x_1$, so

$$\mathbf{x}_+ = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for any constant $x_1 \neq 0$.
The eigenvector $\mathbf{x}_-$ associated with $\lambda_- = 0$ satisfies

$$\mathbf{0} = (M - \lambda_- I)\mathbf{x}_- = \begin{bmatrix} 1 & 0.5 \\ 1 & 0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Each of the equations represented above implies $x_2 = -2x_1$, so

$$\mathbf{x}_+ = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -2x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

for any constant $x_1 \neq 0$.

(c) The population will not go extinct since one of the eigenvalues, namely $\lambda_+ = 1.5$, has absolute value greater than one.