MTH 370, Fall 2009
Solutions to Homework 12

Instructions: Do these calculations by hand (you may use a computer or calculator for simple arithmetic and function evaluations) and show your work.

1. Show that the two-species competition model

$$\frac{dx}{dt} = r_1 x \left(1 - \frac{x + \beta_{12} y}{\kappa_1}\right),$$
$$\frac{dy}{dt} = r_2 y \left(1 - \frac{y + \beta_{21} x}{\kappa_2}\right),$$

has no limit-cycle solutions in the positive quadrant (i.e., when $x > 0$, $y > 0$). [Hint: Set $h(x, y) = \frac{1}{xy}$ and use Dulac's negative criterion.]

Solution: Notice that

$$hf = r_1 \left(\frac{1}{y} - \frac{x + \beta_{12} y}{\kappa_1 y}\right), \quad hg = r_2 \left(\frac{1}{x} - \frac{y + \beta_{21} x}{\kappa_2 x}\right),$$

hence

$$\frac{\partial(hf)}{\partial x} = -\frac{r_1}{\kappa_1 y}, \quad \frac{\partial(hg)}{\partial y} = -\frac{r_2}{\kappa_2 x},$$

and so

$$\frac{\partial(hf)}{\partial x} + \frac{\partial(hg)}{\partial y} = -\left(\frac{r_1}{\kappa_1 y} + \frac{r_2}{\kappa_2 x}\right) < 0 \quad \text{for all } x, y > 0.$$

Thus by Dulac's negative criterion there cannot be any limit cycle solutions in the positive quadrant.

2. Consider the following nondimensional model from the last homework:

$$\frac{du}{d\tau} = c - u + u^2 v,$$
$$\frac{dv}{d\tau} = d - u^2 v.$$

Assuming that $0 < c \ll d$, argue that this system undergoes a Hopf bifurcation when $d \approx 1$.

Solution: Recall that the positive equilibrium is

$$u^* = c + d, \quad v^* = \frac{d}{(c+d)^2},$$

that the Jacobian at this equilibrium is

$$J(u^*, v^*) = \begin{bmatrix} \frac{2d}{c+d} - 1 & (c+d)^2 \\ -\frac{2d}{c+d} & -(c+d)^2 \end{bmatrix},$$

and that the trace and determinant of this Jacobian are

$$\operatorname{tr}(J) = \frac{2d}{c+d} - 1 - (c+d)^2 = \frac{-c+d}{c+d} - (c+d)^2, \quad \det(J) = (c+d)^2.$$

If $c \ll d$ then

$$\operatorname{tr}(J) \approx 1 - d^2, \quad \det(J) \approx d^2,$$

and the eigenvalues of the Jacobian are approximately

$$\lambda_\pm \approx \frac{1 - d^2 \pm \sqrt{(1-d^2)^2 - 4d^2}}{2}.$$

Therefore, there is the possibility of a Hopf bifurcation at $d^2 \approx 1$ (i.e., $d \approx 1$) since in this case the discriminant $\lambda_\pm$ is negative and the real parts of $\lambda_\pm$ are approximately zero. We expect to see limit cycles when $d \lesssim 1$ since this is when the real parts of $\lambda_\pm$ are positive.