MTH 370, Fall 2009
Solutions to Homework 11

Instructions: Do these calculations by hand (you may use a computer or calculator for simple arithmetic and function evaluations) and show your work.

1. Consider the following reactions:

$$X \underset{k_{-1}}{\overset{k_1}{\rightleftarrows}} A, \quad B \xrightarrow{k_2} Y, \quad 2X + Y \xrightarrow{k_3} 3X$$

   (a) Write down the mass action equations for these reactions, treating the concentrations of A and B as positive constants.

   (b) Show that, by making the change of variables

$$u = \sqrt{\frac{k_3}{k_1}}x, \quad v = \sqrt{\frac{k_3}{k_1}}y, \quad \tau = k_1 t,$$

   the mass action equations of part (a) become

$$\begin{aligned} \frac{du}{d\tau} &= c - u + u^2 v \\ \frac{dv}{d\tau} &= d - u^2 v \end{aligned} \tag{1}$$

   where c and d are positive constants.

   (c) Show that the system (1) has exactly one equilibrium, that this equilibrium is positive, and that it is repelling if and only if

$$2d > (c+d)(1+(c+d)^2). \tag{2}$$

   (d) Assuming that the inequality (2) holds, show that the region D bounded by the four lines

$$u = c, \quad v = 0, \quad v = \frac{d}{c^2}, \quad v = \frac{d}{c^2} + c + d - u,$$

   is a trapping region for the solutions of (1).

   (e) Conclude that the region D contains a limit cycle when the inequality (2) holds.

Solutions:

(a)

$$\begin{aligned} \frac{dx}{dt} &= k_{-1}a - k_1 x + k_3 x^2 y \\ \frac{dy}{dt} &= k_2 b - k_3 x^2 y \end{aligned}$$

(b)

$$\frac{du}{dt} = \sqrt{\frac{k_3}{k_1}}\frac{1}{k_1}\frac{dx}{dt} = \frac{k_{-1}}{k_1}\sqrt{\frac{k_3}{k_1}}a - \sqrt{\frac{k_3}{k_1}}x + \left(\sqrt{\frac{k_3}{k_1}}\right)^3 x^2 y = c - u + u^2 v$$

$$\frac{dv}{dt} = \sqrt{\frac{k_3}{k_1}}\frac{1}{k_1}\frac{dy}{dt} = \frac{k_2}{k_1}\sqrt{\frac{k_3}{k_1}}b - \left(\sqrt{\frac{k_3}{k_1}}\right)^3 x^2 y = d - u^2 v$$

(c) The u- and v-nullclines are, respectively,

$$v = \frac{u-c}{u^2}, \quad v = \frac{d}{u^2}.$$

These intersect only at

$$u^* = c+d, \quad v^* = \frac{d}{(c+d)^2},$$

which is positive. The Jacobian of (1) is

$$J(u,v) = \begin{bmatrix} 2uv-1 & u^2 \\ -2uv & -u^2 \end{bmatrix},$$

and so

$$J(u^*,v^*) = \begin{bmatrix} \frac{2d}{c+d}-1 & (c+d)^2 \\ -\frac{2d}{c+d} & -(c+d)^2 \end{bmatrix} \quad \Rightarrow \quad \operatorname{tr}(J) = \frac{2d}{c+d} - 1 - (c+d)^2, \ \det(J) = (c+d)^2.$$

Note that $\det(J) > 0$, and so (u^*, v^*) is repelling if and only if $\operatorname{tr}(J) > 0$, which implies (2).

(d) Let $\mathbf{n}$ be the inward normal vector to the region D. Hence

$$\mathbf{n} = \begin{cases} (1,0)^T, & u = c, \\ (0,1)^T, & v = 0, \\ (0,-1)^T, & v = \frac{d}{c^2}, \\ (-1,-1)^T, & v = \frac{d}{c^2} + c + d - u. \end{cases}$$

If the dot product of a solution's tangent vector,

$$\mathbf{f}(u,v) = \begin{bmatrix} c - u + u^2 v \\ d - u^2 v \end{bmatrix},$$

at a point on the boundary of D with this inward normal vector is ≥ 0, then the solution does not leave the region D transverse to the boundary at that point. We check

$$\begin{aligned} \text{on } u = c: &\quad \mathbf{n} \cdot \mathbf{f}(c,v) = u^2 v \geq 0 \quad (\text{when } v \geq 0) \\ \text{on } v = 0: &\quad \mathbf{n} \cdot \mathbf{f}(u,0) = d > 0 \\ \text{on } v = \frac{d}{c^2}: &\quad \mathbf{n} \cdot \mathbf{f}(u, d/c^2) = d\left(\frac{u^2}{c^2} - 1\right) \geq 0 \quad (\text{when } u \geq c) \\ \text{on } v = \frac{d}{c^2} + c + d - u: &\quad \mathbf{n} \cdot \mathbf{f}(u, d/c^2 + c + d - u) = u - c - d \geq 0 \quad (\text{when } u \geq c + d) \end{aligned}$$

Note that each of the above conditions holds on the boundary, and so D is a trapping region.

(e) We have just shown that D is a trapping region. It is not hard to see that D contains (u^*, v^*), so when this equilbrium is repelling, it follows from the Poincaré-Bendixson theorem that D must contain a limit cycle.