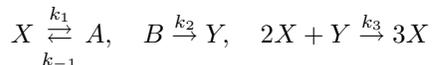


**MTH 370, Fall 2009**  
**Solutions to Homework 11**

**Instructions:** Do these calculations by hand (you may use a computer or calculator for simple arithmetic and function evaluations) and show your work.

1. Consider the following reactions:



- (a) Write down the mass action equations for these reactions, treating the concentrations of  $A$  and  $B$  as positive constants.
- (b) Show that, by making the change of variables

$$u = \sqrt{\frac{k_3}{k_1}}x, \quad v = \sqrt{\frac{k_3}{k_1}}y, \quad \tau = k_1t,$$

the mass action equations of part (a) become

$$\begin{aligned} \frac{du}{d\tau} &= c - u + u^2v \\ \frac{dv}{d\tau} &= d - u^2v \end{aligned} \tag{1}$$

where  $c$  and  $d$  are positive constants.

- (c) Show that the system (1) has exactly one equilibrium, that this equilibrium is positive, and that it is repelling if and only if

$$2d > (c + d)(1 + (c + d)^2). \tag{2}$$

- (d) Assuming that the inequality (2) holds, show that the region  $D$  bounded by the four lines

$$u = c, \quad v = 0, \quad v = \frac{d}{c^2}, \quad v = \frac{d}{c^2} + c + d - u,$$

is a trapping region for the solutions of (1).

- (e) Conclude that the region  $D$  contains a limit cycle when the inequality (2) holds.

**Solutions:**

(a)

$$\begin{aligned} \frac{dx}{dt} &= k_{-1}a - k_1x + k_3x^2y \\ \frac{dy}{dt} &= k_2b - k_3x^2y \end{aligned}$$

(b)

$$\begin{aligned} \frac{du}{d\tau} &= \sqrt{\frac{k_3}{k_1}} \frac{1}{k_1} \frac{dx}{dt} = \frac{k_{-1}}{k_1} \sqrt{\frac{k_3}{k_1}} a - \sqrt{\frac{k_3}{k_1}} x + \left( \sqrt{\frac{k_3}{k_1}} \right)^3 x^2y = c - u + u^2v \\ \frac{dv}{d\tau} &= \sqrt{\frac{k_3}{k_1}} \frac{1}{k_1} \frac{dy}{dt} = \frac{k_2}{k_1} \sqrt{\frac{k_3}{k_1}} b - \left( \sqrt{\frac{k_3}{k_1}} \right)^3 x^2y = d - u^2v \end{aligned}$$

(c) The  $u$ - and  $v$ -nullclines are, respectively,

$$v = \frac{u-c}{u^2}, \quad v = \frac{d}{u^2}.$$

These intersect only at

$$u^* = c+d, \quad v^* = \frac{d}{(c+d)^2},$$

which is positive. The Jacobian of (1) is

$$J(u, v) = \begin{bmatrix} 2uv - 1 & u^2 \\ -2uv & -u^2 \end{bmatrix},$$

and so

$$J(u^*, v^*) = \begin{bmatrix} \frac{2d}{c+d} - 1 & (c+d)^2 \\ -\frac{2d}{c+d} & -(c+d)^2 \end{bmatrix} \Rightarrow \operatorname{tr}(J) = \frac{2d}{c+d} - 1 - (c+d)^2, \quad \det(J) = (c+d)^2.$$

Note that  $\det(J) > 0$ , and so  $(u^*, v^*)$  is repelling if and only if  $\operatorname{tr}(J) > 0$ , which implies (2).

(d) Let  $\mathbf{n}$  be the inward normal vector to the region  $D$ . Hence

$$\mathbf{n} = \begin{cases} (1, 0)^T, & u = c, \\ (0, 1)^T, & v = 0, \\ (0, -1)^T, & v = \frac{d}{c^2}, \\ (-1, -1)^T, & v = \frac{d}{c^2} + c + d - u. \end{cases}$$

If the dot product of a solution's tangent vector,

$$\mathbf{f}(u, v) = \begin{bmatrix} c - u + u^2v \\ d - u^2v \end{bmatrix},$$

at a point on the boundary of  $D$  with this inward normal vector is  $\geq 0$ , then the solution does not leave the region  $D$  transverse to the boundary at that point. We check

$$\text{on } u = c: \quad \mathbf{n} \cdot \mathbf{f}(c, v) = u^2v \geq 0 \quad (\text{when } v \geq 0)$$

$$\text{on } v = 0: \quad \mathbf{n} \cdot \mathbf{f}(u, 0) = d > 0$$

$$\text{on } v = \frac{d}{c^2}: \quad \mathbf{n} \cdot \mathbf{f}(u, d/c^2) = d \left( \frac{u^2}{c^2} - 1 \right) \geq 0 \quad (\text{when } u \geq c)$$

$$\text{on } v = \frac{d}{c^2} + c + d - u: \quad \mathbf{n} \cdot \mathbf{f}(u, d/c^2 + c + d - u) = u - c - d \geq 0 \quad (\text{when } u \geq c + d)$$

Note that each of the above conditions holds on the boundary, and so  $D$  is a trapping region.

(e) We have just shown that  $D$  is a trapping region. It is not hard to see that  $D$  contains  $(u^*, v^*)$ , so when this equilibrium is repelling, it follows from the Poincaré-Bendixson theorem that  $D$  must contain a limit cycle.