MTH 370, Fall 2009
Solutions to Homework 10

Instructions: Do these calculations by hand (you may use a computer or calculator for simple arithmetic and function evaluations) and show your work.

1. In the real world, trimolecular reactions are rare, although trimers are not. Consider the following trimerization reaction in which three monomers of X combine to form the trimer Z:

$$X + X \underset{k_{-1}}{\overset{k_1}{\rightleftarrows}} Y$$

$$X + Y \underset{k_{-2}}{\overset{k_2}{\rightleftarrows}} Z$$

   (a) Write down the mass-action equations for these reactions.

   (b) Show that
$$\frac{dx}{dt} + 2\frac{dy}{dt} + 3\frac{dz}{dt} = 0.$$
   Why does this equation hold?

   (c) Letting x_0 denote the initial concentration of monomers, suppose
$$k_{-1} \gg k_{-2},\ k_{-1} \gg k_2 x_0.$$
   Use a quasi-steady-state approximation to find the rate of production of Z, and show that it is proportional to x^3.

Solutions:

(a)

$$\frac{dx}{dt} = -2k_1x^2 - k_2xy + 2k_{-1}y + k_{-2}z$$
$$\frac{dy}{dt} = k_1x^2 + k_{-2}z - k_2xy - k_{-1}y$$
$$\frac{dz}{dt} = k_2xy - k_{-2}z$$

(b) This is equation represents the conservation of total monomer concentration.

(c) Setting

$$a = \frac{x}{x_0}, \quad b = \frac{y}{x_0}, \quad c = \frac{z}{x_0}, \quad \tau = k_2x_0t, \quad \lambda = \frac{k_{-2}}{k_2x_0}, \quad \mu = \frac{k_1x_0}{k_{-1}}, \quad \epsilon = \frac{k_2x_0}{k_{-1}},$$

the above equations can be written

$$\epsilon\frac{da}{d\tau} = \epsilon(\lambda c - ab) - 2\mu a^2 + 2b$$
$$\epsilon\frac{db}{d\tau} = \epsilon(\lambda c - ab) + \mu a^2 - b$$
$$\frac{dc}{d\tau} = ab - \lambda c$$

Setting $\epsilon = 0$ gives the "slow" equations in which a and b have already reached quasi-equilibrium, and during which
$$b = \mu a^2.$$
Substituting this into the equation for c yields

$$\frac{dc}{d\tau} = \mu a^3 - \lambda c.$$

2. The length of a microtubule changes by a process called *treadmilling* – monomer is added to one end of the microtubule and removed from the other. A simple set of reactions for this process is the following. Let X_n denote a microtubule comprised of n monomers. We say that n is the *length* of such a microtubule. Then for each $n \geq 1$,

$$X_1 + X_n \underset{k_-}{\overset{k_+}{\rightleftarrows}} X_{n+1},$$

where k_+ is the rate of adding monomer, and k_- is the rate of removing it. Assuming that there is a total concentration c of monomers, find the equilibrium distribution of microtubule lengths.

Solutions: The mass-action equations are

$$\begin{aligned}\frac{dx_1}{dt} &= -k_+x_1\left(\sum_{n=1}^{\infty} x_n + x_1\right) + k_-\left(\sum_{n=2}^{\infty} x_n + x_2\right),\\ \frac{dx_n}{dt} &= k_+x_1x_{n-1} - (k_+x_1 + k_-)x_n + k_-x_{n+1} \quad (n > 1).\end{aligned}$$

It is "easy" to check that the following conservation equation holds,

$$\sum_{n=1}^{\infty} nx_n = c. \tag{1}$$

At equilibrium, the equation above for $n > 1$ can be written

$$0 = \mu x_{n-1} - (1+\mu)x_n + x_{n+1} \qquad \left(\mu = \frac{k_+x_1}{k_-}\right)$$

which is a second-order linear difference equations for x_n. Setting $x_n = \lambda^n$, the above equation is reduced to a quadratic equation for λ,

$$0 = \mu - (1+\mu)\lambda + \lambda^2,$$

whose solutions are

$$\lambda_\pm = \frac{1+\mu \pm \sqrt{(1+\mu)^2 - 4\mu}}{2} = \frac{1+\mu \pm \sqrt{(1-\mu)^2}}{2} \Rightarrow \lambda_+ = 1,\ \lambda_- = \mu.$$

Therefore, the general form of x_n is

$$x_n = c_1 + c_2\mu^n$$

for some constants c_1 and c_2 to be determined. From (1) it is clear that $c_1 = 0$ and $\mu < 1$ else the sum would be infinite. Substituting $x_n = c_2\mu^n$ into (1) yields

$$c = c_2\sum_{n=1}^{\infty} n\mu^n = c_2\mu\frac{d}{d\mu}\left(\sum_{n=1}^{\infty}\mu^n\right) = c_2\mu\frac{d}{d\mu}\left(\frac{1}{1-\mu} - 1\right) = \frac{c_2\mu}{(1-\mu)^2} \Rightarrow c_2 = \frac{c(1-\mu)^2}{\mu}.$$

Therefore,

$$x_n = c(1-\mu)^2\mu^{n-1} \quad (n \geq 1). \tag{2}$$

Recall that $\mu = k_+x_1/k_-$, hence setting $n = 1$ in the equation above and multiplying by k_+/k_- yields

$$\mu = \frac{ck_+}{k_-}(1-\mu)^2 \Rightarrow 0 = 1 - \left(2 + \frac{k_-}{ck_+}\right)\mu + \mu^2,$$

so

$$\mu_\pm = 1 + \frac{k_-}{2ck_+} \pm \sqrt{\left(1 + \frac{k_-}{2ck_+}\right)^2 - 1}.$$

Since

$$\sqrt{\left(1+\frac{k_-}{2ck_+}\right)^2 - 1} = \sqrt{\frac{k_-}{ck_+} + \left(\frac{k_-}{2ck_+}\right)^2} > \frac{k_-}{2ck_+},$$

we find $\mu_+ > 1$ but $\mu_- < 1$. Thus $\mu = \mu_-$ in (2).

Of course, x_n is only the equilibrium concentration of microtubules of length n. To find the distribution, we must first calculate the total concentration of microtubules,

$$C = \sum_{n=1}^{\infty} x_n = c(1-\mu)^2 \sum_{n=1}^{\infty} \mu^n = c(1-\mu)^2 \frac{\mu}{1-\mu} = c\mu(1-\mu)$$

and then divide x_n by C,

$$p_n = \frac{x_n}{C} = (1-\mu)\mu^{n-2}.$$

Therefore, the lengths are distributed geometrically. The number p_n is the probability of finding a microtubule of length n amongst all the microtubules at equilibrium.