MTH 370, Fall 2009
Solutions to Homework 9

Instructions: Do these calculations by hand (you may use a computer or calculator for simple arithmetic and function evaluations) and show your work.

1. Write down the mass-action equations for the following chemical reactions:

   (a) $X + Y \xrightarrow{k} 2Y$

   (b) $2X \xrightarrow{k} X + Y$

   (c) $2X + Y \xrightarrow{k} X + 2Y$

   Solutions:

   (a)
$$\frac{dx}{dt} = -kxy$$
$$\frac{dy}{dt} = kxy$$

   (b)
$$\frac{dx}{dt} = -kx^2$$
$$\frac{dy}{dt} = kx^2$$

   (c)
$$\frac{dx}{dt} = -kx^2y$$
$$\frac{dy}{dt} = kx^2y$$

2. Consider the following chemical reactions:
$$A + X \xrightarrow{k_1} 2X$$
$$X + Y \xrightarrow{k_2} 2Y$$
$$Y \xrightarrow{k_3} \emptyset$$

   Assuming the concentration of A is kept constant, show that the mass-action equations for these reactions are the same as the Loktka-Volterra model for predator-prey systems. Which chemical is the "prey" and which is the "predator"?

   Solutions:
$$\frac{dx}{dt} = (k_1a - k_2y)x$$
$$\frac{dy}{dt} = (k_2x - k_3)y$$

   X is the prey, Y is the predator.

3. Consider the following chemical reactions:
$$X \underset{\beta}{\overset{2\alpha}{\rightleftarrows}} Y \underset{2\beta}{\overset{\alpha}{\rightleftarrows}} Z$$

(a) Write down the mass-action equations for the concentrations x, y and z, and show that $x+y+z$ is constant.

(b) Show that when $x+y+z=1$, the steady-state solution of the mass-action equations is a binomial distribution with parameter $p=\alpha/(\alpha+\beta)$.

(c) What kind of biological system might these reactions describe?

Solutions:

(a)

$$\begin{aligned}\frac{dx}{dt} &= -2\alpha x + \beta y \\ \frac{dy}{dt} &= 2\alpha x - (\alpha+\beta)y + 2\beta z \\ \frac{dz}{dt} &= \alpha y - 2\beta z\end{aligned}$$

Summing these equations together we get

$$\frac{d(x+y+z)}{dt} = 0 \quad \Rightarrow \quad x+y+z = \text{constant}$$

(b) In the first and third equations above, set the derivatives to zero and $y = 1-x-z$ to get

$$\begin{aligned}0 &= -2\alpha x^* + \beta(1-x^*-z^*) \\ 0 &= \alpha(1-x^*-z^*) - 2\beta z^*\end{aligned} \quad \Rightarrow \quad \begin{aligned}(2\alpha+\beta)x^* + \beta z^* &= \beta \\ \alpha x^* + (\alpha+2\beta)z^* &= \alpha\end{aligned}.$$

Solving these yields

$$x^* = \left(\frac{\beta}{\alpha+\beta}\right)^2 = (1-p)^2, \quad z^* = \left(\frac{\alpha}{\alpha+\beta}\right)^2 = p^2,$$
$$y^* = 1-x^*-z^* = \frac{2\alpha\beta}{(\alpha+\beta)^2} = 2p(1-p),$$

hence this is a binomial distribution with parameter p (or $1-p$).

(c) For example, these equations models the kinetics of an ion channel comprised of two identical, independent subunits which open at a rate α and close at a rate β.