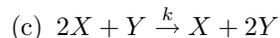
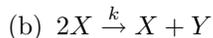
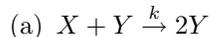


MTH 370, Fall 2009
Solutions to Homework 9

Instructions: Do these calculations by hand (you may use a computer or calculator for simple arithmetic and function evaluations) and show your work.

1. Write down the mass-action equations for the following chemical reactions:



Solutions:

(a)

$$\begin{aligned}\frac{dx}{dt} &= -kxy \\ \frac{dy}{dt} &= kxy\end{aligned}$$

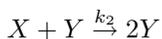
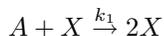
(b)

$$\begin{aligned}\frac{dx}{dt} &= -kx^2 \\ \frac{dy}{dt} &= kx^2\end{aligned}$$

(c)

$$\begin{aligned}\frac{dx}{dt} &= -kx^2y \\ \frac{dy}{dt} &= kx^2y\end{aligned}$$

2. Consider the following chemical reactions:



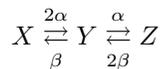
Assuming the concentration of A is kept constant, show that the mass-action equations for these reactions are the same as the Lotka-Volterra model for predator-prey systems. Which chemical is the “prey” and which is the “predator”?

Solutions:

$$\begin{aligned}\frac{dx}{dt} &= (k_1a - k_2y)x \\ \frac{dy}{dt} &= (k_2x - k_3)y\end{aligned}$$

X is the prey, Y is the predator.

3. Consider the following chemical reactions:



- (a) Write down the mass-action equations for the concentrations x , y and z , and show that $x + y + z$ is constant.
- (b) Show that when $x + y + z = 1$, the steady-state solution of the mass-action equations is a binomial distribution with parameter $p = \alpha/(\alpha + \beta)$.
- (c) What kind of biological system might these reactions describe?

Solutions:

(a)

$$\begin{aligned}\frac{dx}{dt} &= -2\alpha x + \beta y \\ \frac{dy}{dt} &= 2\alpha x - (\alpha + \beta)y + 2\beta z \\ \frac{dz}{dt} &= \alpha y - 2\beta z\end{aligned}$$

Summing these equations together we get

$$\frac{d(x + y + z)}{dt} = 0 \quad \Rightarrow \quad x + y + z = \text{constant}$$

(b) In the first and third equations above, set the derivatives to zero and $y = 1 - x - z$ to get

$$\begin{aligned}0 &= -2\alpha x^* + \beta(1 - x^* - z^*) & \Rightarrow & \quad (2\alpha + \beta)x^* + \beta z^* = \beta \\ 0 &= \alpha(1 - x^* - z^*) - 2\beta z^* & \Rightarrow & \quad \alpha x^* + (\alpha + 2\beta)z^* = \alpha\end{aligned}$$

Solving these yields

$$\begin{aligned}x^* &= \left(\frac{\beta}{\alpha + \beta}\right)^2 = (1 - p)^2, & z^* &= \left(\frac{\alpha}{\alpha + \beta}\right)^2 = p^2, \\ y^* &= 1 - x^* - z^* = \frac{2\alpha\beta}{(\alpha + \beta)^2} = 2p(1 - p),\end{aligned}$$

hence this is a binomial distribution with parameter p (or $1 - p$).

(c) For example, these equations models the kinetics of an ion channel comprised of two identical, independent subunits which open at a rate α and close at a rate β .