MTH 370, Fall 2009
Solutions to Homework 8

Instructions: Do these calculations by hand (you may use a computer or calculator for simple arithmetic and function evaluations) and show your work.

1. One unrealistic feature of the the Lotka-Volterra model is that it the prey population grows without bound in the absence of predators. We can remedy this fault by introducing a logistic-type term for the prey's reproduction rate:

$$\frac{dx}{dt} = (a(K-x) - by)x,$$
$$\frac{dy}{dt} = (dx - c)y,$$

where $K > 0$ is the carrying capacity for the prey population. Analyze this modified Lotka-Volterra model the same way we did the original Lotka-Volterra model in class. That is, find the equilibria and determine their stability type, find the x– and y–nullclines and determine in which direction solutions traverse them, and then plot all this information, including a few representative solutions, in the phase–plane.

Solutions: The equilibria satisfy

$$0 = (a(K-x^*) - by^*)x^*,$$
$$0 = (dx^* - c)y^*.$$

The second equation implies that either $y^* = 0$ or $x^* = c/d$.

If $y^* = 0$, then the first equation becomes

$$0 = a(K-x^*)x^* \Rightarrow x^* = 0 \text{ or } K,$$

so $(x^*, y*) = (0,0)$ and $(x^*, y^*) = (K, 0)$ are both equilibria.

If $y^* \neq 0$, then $x^* = c/d$ and the first equation becomes

$$0 = \left(a\left(K - \frac{c}{d}\right) - by^*\right)\frac{c}{d} \Rightarrow y^* = \frac{a}{b}\left(K - \frac{c}{d}\right),$$

hence $(x^*, y^*) = \left(\frac{c}{d}, \frac{a}{b}\left(K - \frac{c}{d}\right)\right)$ is also an equilibrium. This equilibrium is only biologically relevant if $Kd > c$, and so we assume this from now on.

The Jacobian is

$$J(x,y) = \begin{bmatrix} a(K-2x) - by & -bx \\ dy & dx - c \end{bmatrix}.$$

At the origin the Jacobian is

$$J(0,0) = \begin{bmatrix} aK & 0 \\ 0 & -c \end{bmatrix},$$

therefore $\lambda_+ = aK > 0$ and $\lambda_- = -c < 0$ and hence the origin is still a saddle node.

At $(x^*, y^*) = (K, 0)$ the Jacobian is

$$J(K,0) = \begin{bmatrix} -aK & -bK \\ 0 & dK - c \end{bmatrix},$$

therefore $\lambda_+ = dK - c > 0$ and $\lambda_- = -aK < 0$ and hence this equilibrium is also a saddle node.

At $(x^*, y^*) = \left(\frac{c}{d}, \frac{a}{b}\left(K - \frac{c}{d}\right)\right)$ the Jacobian is

$$J\left(\frac{c}{d}, \frac{a}{b}\left(K - \frac{c}{d}\right)\right) = \begin{bmatrix} -\frac{ac}{d} & -\frac{bc}{d} \\ \frac{ad}{b}\left(K - \frac{c}{d}\right) & 0 \end{bmatrix}.$$

Since $\mathrm{tr}(J) = -ac/d < 0$ and $\det(J) = ac(K - c/d) > 0$, we know that the real part of the eigenvalues are both negative, hence this equilibrium is stable. Note that without further information we cannot determine if this equilibrium is a stable node or a stable focus since

$$\lambda_{\pm} = \frac{-ac/d \pm \sqrt{(ac/d)^2 - 4ac(K - c/d)}}{2}$$

and the discriminant can be either positive or negative.