MTH 370, Fall 2009
Solutions to Homework 7

Instructions: Do these calculations by hand (you may use a computer or calculator for simple arithmetic and function evaluations) and show your work.

1. Solve the following systems of first-order linear ODEs. In each problem, classify the type and stability of the origin.

   (a) $\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$

   (b) $\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad A = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}$

   (c) $\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

   Solutions:

   (a) The eigenvalues of A are 1 and 4, so the origin is an unstable node.

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) = \begin{bmatrix} (2x(0) - y(0))e^t + (x(0) + y(0))e^{4t} \\ -(2x(0) - y(0))e^t + 2(x(0) + y(0))e^{4t} \end{bmatrix}$$
$$= (2x(0) - y(0))e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + (x(0) + y(0))e^{4t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

   (b) The eigenvalues of A are $-2 + i$ and $-2 - i$, so the origin is a stable focus.

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) = e^{-2t} \begin{bmatrix} x(0)\cos(t) - y(0)\sin(t) \\ x(0)\sin(t) + y(0)\cos(t) \end{bmatrix}.$$

   (c) The eigenvalues of A are $\frac{5+\sqrt{33}}{2} > 0$ and $\frac{5-\sqrt{33}}{2} < 0$, so the origin is a saddle node. Solutions way too ugly...

2. Consider the system of first-order linear ODEs

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{1}$$

   (a) Solve (1) first by solving the second equation, and then plugging this into the first equation and solving it by integrating factors.

   (b) Now try to solve (1) by calculating e^{At}. If you have trouble, explain why. Can you calculate e^{At} directly from its Taylor series?

   Solutions:

   (a) The second equation is

$$\frac{dy}{dt} = -y \Rightarrow y(t) = e^{-t}y(0) = e^{-t}$$

   The first equation is then

$$\frac{dx}{dt} = -x + y = -x + e^{-t} \Rightarrow x(t) = e^{-t}(t + x(0)) = e^{-t}(t + 1) \text{ using an integration factor.}$$

   Therefore, the solution is

$$\mathbf{x}(t) = e^{-t} \begin{bmatrix} t + 1 \\ 1 \end{bmatrix}.$$

(b) The matrix A has a double eigenvalue $\lambda = -1$, however we can only find one eigenvector

$$\mathbf{0} = (A - \lambda I)\mathbf{v} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v+2 \end{bmatrix} \Rightarrow \mathbf{v} = v_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

so we can't use the method from class. Notice however that

$$(At)^n = \begin{bmatrix} (-t)^n & -n(-t)^n \\ 0 & (-t)^n \end{bmatrix},$$

hence

$$e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{bmatrix} (-t)^n & -n(-t)^n \\ 0 & (-t)^n \end{bmatrix} = \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix} = e^{-t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix},$$

so the solution is

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) = e^{-t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = e^{-t} \begin{bmatrix} t+1 \\ 1 \end{bmatrix}.$$