MTH 370, Fall 2009
Solutions to Homework 5

Instructions: Do these calculations by hand (you may use a computer or calculator for simple arithmetic and function evaluations) and show your work.

1. Suppose that a population of hosts and parasitoids follow the Nicholson-Bailey equations except that in each generation a fraction $p < 1$ of the hosts have a safe refuge from attack. Thus the equations become

$$x_{n+1} = re^{-ay_n}(1-p)x_n + rpx_n$$
$$y_{n+1} = cr(1-e^{-ay_n})(1-p)x_n$$

(a) Find the fixed points and determine their stability. You may assume that $rp \neq 1$.

(b) Can this strategy of the hosts stabilize the nonzero fixed point?

Solution: The fixed points are

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\ln\left(\frac{r-rp}{1-rp}\right)}{ac(r-1)} \\ \frac{\ln\left(\frac{r-rp}{1-rp}\right)}{a} \end{bmatrix}$$

and the Jacobian is

$$J(x,y) = \begin{bmatrix} r((1-p)e^{-ay}+p) & -ar(1-p)e^{-ay}x \\ cr(1-p)(1-e^{-ay}) & acr(1-p)e^{-ay}x \end{bmatrix}.$$

Hence

$$J(0,0) = \begin{bmatrix} r & 0 \\ 0 & 0 \end{bmatrix}, \quad J\left(\frac{\ln\left(\frac{r-rp}{1-rp}\right)}{ac(r-1)}, \frac{\ln\left(\frac{r-rp}{1-rp}\right)}{a}\right) = \begin{bmatrix} 1 & -\frac{1-rp}{c(r-1)}\ln\left(\frac{r-rp}{1-rp}\right) \\ c(r-1) & \frac{1-rp}{r-1}\ln\left(\frac{r-rp}{1-rp}\right) \end{bmatrix}.$$

Notice that the second fixed point is both nonzero and nonnegative only when $r > 1$ and $rp < 1$, so we must require this.

Just as with the original Nicholson-Bailey model, the extinction fixed point is unstable since the eigenvalues are $r > 1$ and 0. For the nonzero fixed point we compute

$$\operatorname{tr}(J) = 1 + \frac{1-rp}{r-1}\ln\left(\frac{r-rp}{1-rp}\right), \quad \det(J) = \frac{r(1-rp)}{r-1}\ln\left(\frac{r-rp}{1-rp}\right).$$

Notice that $\operatorname{tr}(J) > 1$ and $\det(J) > 0$ for all $r > 1$ and $rp < 1$, and if $rp \approx 1$ then $\operatorname{tr}(J) \approx 1$ and $\det(J) \approx 0$. This means that the eigenvalues of J are real since in this case the discriminant $\operatorname{tr}(J)^2 - 4\det(J) \approx 1$ is positive. One can show (see pages 57-58 of Edelstein-Keshet's book) that when the eigenvalues of J are real, they both have absolute values less than one if and only if

$$|\operatorname{tr}(J)| < \min(2, 1+\det(J)).$$

which holds in this case since $\operatorname{tr}(J) \approx 1$, and

$$1+\det(J) = 1 + \frac{r(1-rp)}{r-1}\ln\left(\frac{r-rp}{1-rp}\right) > 1 + \frac{1-rp}{r-1}\ln\left(\frac{r-rp}{1-rp}\right) = \operatorname{tr}(J)$$

since $r > 1$.

Therefore, if we choose p large enough so that $rp < 1$ and $rp \approx 1$, then the nonzero fixed point is stable.