MTH 370, Fall 2009
Solutions to Homework 4

Instructions: Do these calculations by hand (you may use a computer or calculator for simple arithmetic and function evaluations) and show your work.

1. Recall from class that the characteristic equation of the matrix A can be written

$$0 = \det(A - \lambda I) = \lambda^2 - \operatorname{tr}(A)\lambda + \det(A).$$

   (a) Letting λ_+ and λ_- denote the two eigenvalues of A, prove that

$$\operatorname{tr}(A) = \lambda_+ + \lambda_- \quad \text{and} \quad \det(A) = \lambda_+\lambda_-.$$

   [Hint: Use the fact that λ_+ and λ_- are the two roots of the quadratic polynomial $\det(A - \lambda I)$.]

   (b) Verify that (a) is indeed true for the matrix

$$A = \begin{bmatrix} 7 & 2 \\ 2 & 7 \end{bmatrix}.$$

   Solution:

   (a) Since λ_+ and λ_- are the roots of the characteristic polynomial, we know

$$\lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = (\lambda - \lambda_+)(\lambda - \lambda_-) = \lambda^2 - (\lambda_+ + \lambda_-)\lambda + \lambda_+\lambda_-.$$

   Equating coefficients gives the result.

   (b) Recall that the eigenvalues of A are $\lambda_+ = 9$ and $\lambda_- = 5$, and in fact $\operatorname{tr}(A) = 7 + 7 = 14 = 9 + 5$ and $\det(A) = 7 \cdot 7 - 2 \cdot 2 = 45 = 9 \cdot 5$.

2. Let

$$A = \begin{bmatrix} 0 & 3 \\ 3 & -1 \end{bmatrix}$$

   (a) Compute the inverse A^{-1} of A.

   (b) Solve the following matrix equation for $\mathbf{x}$:

$$A\mathbf{x} = \mathbf{b} = \begin{bmatrix} -9 \\ 18 \end{bmatrix}$$

   Solution:

   (a) Using the formula from class

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{if and only if} \quad A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix},$$

   we compute $\det(A) = 0(-1) - 3^2 = -9$ and

$$A^{-1} = -\frac{1}{9} \begin{bmatrix} -1 & -3 \\ -3 & 0 \end{bmatrix}.$$

   (b)

$$\mathbf{x} = A^{-1}\mathbf{b} = -\frac{1}{9} \begin{bmatrix} -1 & -3 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} -9 \\ 18 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}.$$

3. Consider the age-structured population model from class,

$$\mathbf{x}_{n+1} = M\mathbf{x}_n \tag{1}$$

where

$$M = \begin{bmatrix} p & q \\ k & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_n = \begin{bmatrix} A_n \\ J_n \end{bmatrix}.$$

(a) Show that when $p = 0.5$, $q = 0.1$ and $k = 2$, the population will go extinct, no matter the initial size of the population.

(b) Setting $p = 0.5$ and $k = 2$, find the critical value q_c such that if $q > q_c$ then the population grows, but if $q < q_c$ then the population goes extinct.

Solution:

(a) Recall from class that the eigenvalues are

$$\lambda_\pm = \frac{p \pm \sqrt{p^2 + 4qk}}{2} \approx 0.762, -0.262.$$

The absolute value of each is less than one, so the solution approaches $(A, J) = (0, 0)$ as $n \to \infty$.

(b) The critical point $q = q_c$ solves the equation

$$1 = |\lambda_+| = \frac{0.5 + \sqrt{0.25 + 8q}}{2} \Rightarrow q_c = 0.25.$$

The reason we only consider λ_+ is that the q that solves $1 = |\lambda_-|$ is larger than q_c. Note that $\lambda_+ > 1$ if and only if $q > q_c$.