MTH 370, Fall 2009
Solutions to Homework 2

1. Does the difference equation

$$c_{n+1} = c_n(3.5 - 35c_n)$$

have a solution of period 2? If so, is it stable?

Solution:

Factor 3.5 out of the RHS of $c_{n+1} = c_n(3.5 - 35c_n)$ to get

$$c_{n+1} = 3.5c_n(1 - 10c_n).$$

Multiply both sides of this equation by 10 to get

$$10c_{n+1} = 3.5(10c_n)(1 - 10c_n).$$

Now set $b_n = 10c_n$ to get

$$b_{n+1} = 3.5b_n(1 - b_n).$$

This is our original difference equation in normal form, hence $r = 3.5$. Recall from class that $r_2 = 3$ and $r_4 \approx 3.45$, hence

$$r_2 < r_4 < 3.5$$

and therefore period-2 solutions exist but are unstable.

2. Consider again the nonlinear difference equation

$$c_{n+1} = rc_n \mathrm{e}^{-c_n}. \tag{1}$$

Find the point r_2 such that solutions of period 2 exist for $r \geq r_2$ and do not exist for $r < r_2$.

Solution:

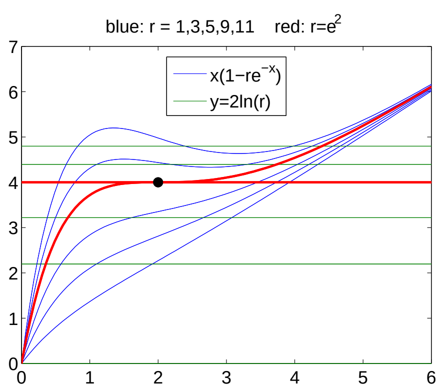
To show that a period-2 solution exists, we need to show that the equation

$$c = f(f(c)) = r^2 c\mathrm{e}^{-c(1+re^{-c})},$$

has four solutions, two of which are the fixed points $c = 0$ and $c = \ln(r)$ of (1) (see Homework 1), and two of which are the two points in the period-2 solution. In (1), either $c = 0$ or

$$2\ln(r) = c(1 + re^{-c}). \tag{2}$$

We know $c = \ln(r)$ is a solution of (2), so we need to show (2) has two more solutions than this. We can do this graphically. In the figure below I have shown the LHS and RHS of (2) plotted for specific values of r. Where these graphs cross is a solution to (2).



The graphs always intersect at $c = \ln(r)$, and at a certain value $r = r_2$ (indicated in red) the function $g(c) = c(1 + re^{-c})$ intersects the horizontal line $2\ln(r)$ only once (so at $c = \ln(r)$), and at this point of intersection the derivative of g is zero. For $r < r_2$ there is only one solution of (2), but for $r > r_2$ there are three solutions. We calculate

$$g'(c) = 1 + (1 - c)re^{-c}$$

hence

$$g'(\ln(r_2)) = 2 - \ln(r_2) = 0 \text{ if and only if } r_2 = e^2.$$

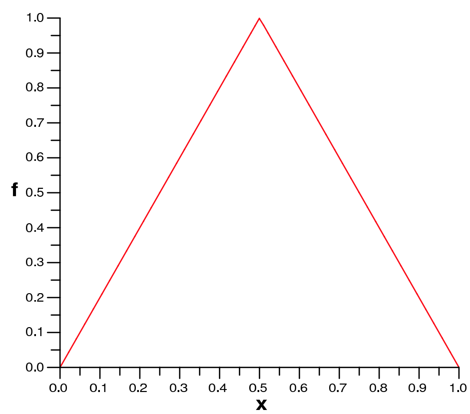
Therefore, a solution of period-2 exists for $r > r_2 = e^2$.

3. Consider the following one-dimensional nonlinear difference equation,

$$x_{n+1} = f(x_n), \tag{3}$$

where f is the *tent map*,

$$f(x) = \begin{cases} 2x, & x \in [0, \frac{1}{2}], \\ 2(1 - x), & x \in (\frac{1}{2}, 1]. \end{cases}$$



(a) Find the fixed points of equation (3) and determine their stability.

(b) Equation (3) has a solution of period two. Find the two points in this solution and determine the solution's stability.

Solution:

(a) $x^* = f(x^*) \Rightarrow x^* = 0, \ x^* = \frac{2}{3}$

$$f'(x) = \begin{cases} 2, & x \in [0, \frac{1}{2}], \\ -2, & x \in (\frac{1}{2}, 1]. \end{cases} \Rightarrow |f'(x^*)| = 2 > 1 \Rightarrow \text{ both fixed points are unstable.}$$

(b)

$$x = f^2(x) = \begin{cases} 4x, & x \in [0, \frac{1}{4}], \\ 4(\frac{1}{2} - x), & x \in (\frac{1}{4}, \frac{1}{2}], \\ 4(x - \frac{1}{2}), & x \in (\frac{1}{2}, \frac{3}{4}], \\ 4(1 - x), & x \in (\frac{3}{4}, 1]. \end{cases} \Rightarrow x^{**} = \frac{2}{5}, \ x^{**} = \frac{4}{5}.$$

$$(f^2)'(x) = \begin{cases} 4, & x \in [0, \frac{1}{4}], \\ -4, & x \in (\frac{1}{4}, \frac{1}{2}], \\ 4, & x \in (\frac{1}{2}, \frac{3}{4}], \\ -4, & x \in (\frac{3}{4}, 1]. \end{cases} \Rightarrow \left|(f^2)'(x^{**})\right| = 4 > 1 \Rightarrow \text{ period-2 solution unstable.}$$