Introduction to Difference Equations

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1 The Difference Equation $\Delta a_n = n^k$

The Take Home exercises are examples of *difference equations*. As you might guess, a difference equation is an equation that contains sequence differences. We *solve* a difference equation by finding a sequence that satisfies the equation, and we call that sequence a *solution* of the equation.

The first three Take Home exercises ask for the solutions to difference equations of the form

$$\Delta a_n = n^k$$

where k is some natural number. As the solutions hinted (and you may have found for yourself), these equations are very easy to solve if we can express n^k in terms of falling factorials. Recall that a falling factorial is defined as

$$n^{\underline{k}} = \frac{n!}{(n-k)!} = n(n-1)(n-2)\cdots(n-k+2)(n-k+1).$$

Thus

$$\begin{aligned}
n^{\underline{0}} &= 1 \\
n^{\underline{1}} &= n \\
n^{\underline{2}} &= n(n-1) = n^2 - n \\
n^{\underline{3}} &= n(n-1)(n-2) = n^3 - 3n^2 + 2n \\
&\vdots
\end{aligned}$$

Using these relationships, we can write

$$\begin{aligned}
1 &= n^{\underline{0}} \\
n &= n^{\underline{1}} \\
n^2 &= n^2 - n + n = n^{\underline{2}} + n^{\underline{1}} \\
n^3 &= n^3 - 3n^2 + 2n + 3(n^2 - n) + n = n^{\underline{3}} + 3n^{\underline{2}} + n^{\underline{1}} \\
&\vdots
\end{aligned}$$

and if we had the time, we could write every monomial n^k as a linear combination of the falling factorials $n^{\underline{0}}, n^{\underline{1}}, \ldots, n^{\underline{k}}$.

How does this help us find a solution to $\Delta a_n = n^k$? Simply because the difference equation $\Delta b_n = n^{\underline{k}}$ is really easy to solve. Remember that

$$\Delta n^{\underline{k+1}} = (k+1)n^{\underline{k}}$$

when $k \geq 0$. Thus

$$b_n = \frac{1}{k+1} n^{\underline{k+1}}$$

is a solution of $\Delta b_n = n^{\underline{k}}$.

So, to solve $\Delta c_n = n$, for example, we just remember that $n = n^{\underline{1}}$ and solve

$$\Delta c_n = n^{\underline{1}}$$

which has

$$c_n = \frac{1}{2} n^{\underline{2}}$$

as a solution.

To solve $\Delta d_n = n^2$, we write $n^2 = n^{\underline{2}} + n^{\underline{1}}$ and solve

$$\Delta d_n = n^{\underline{2}} + n^{\underline{1}}$$

which has

$$d_n = \frac{1}{3} n^{\underline{3}} + \frac{1}{2} n^{\underline{2}}$$

as a solution.

To solve $\Delta e_n = n^3$, we write $n^3 = n^{\underline{3}} + 3n^{\underline{2}} + n^{\underline{1}}$ and solve

$$\Delta e_n = n^{\underline{3}} + 3n^{\underline{2}} + n^{\underline{1}}$$

which has

$$e_n = \frac{1}{4} n^{\underline{4}} + n^{\underline{3}} + \frac{1}{2} n^{\underline{2}}$$

as a solution.

You may be wondering whether or not these are the only solutions to these difference equations. In fact, they are not. To find out what the other solutions are, we need to a few results.

Lemma 1. *The only solutions to the difference equation $\Delta a_n = 0$ are the constant sequences $a_n = c$ for some number c.*

Proof. $\Delta a_n = 0$ means $a_{n+1} - a_n = 0$, or $a_{n+1} = a_n$, for all n. Thus $a_0 = a_1 = a_2 = \cdots = a_n = \cdots$. □

Theorem 1. *Let a_n and b_n be sequences such that $\Delta a_n = \Delta b_n$. Then $a_n = b_n + c$ for some number c.*

Proof. If $\Delta a_n = \Delta b_n$, then by the linearity of Δ,

$$0 = \Delta a_n - \Delta b_n = \Delta(a_n - b_n).$$

So the difference of the sequence $a_n - b_n$ is zero. By Lemma 1, $a_n - b_n$ is a constant sequence; i.e. $a_n - b_n = c$ for some number c. This implies $a_n = b_n + c$. $\square$

This theorem is *really* important and useful. It tells us that if we know just one solution of $\Delta a_n = z_n$, we actually know all of the solutions, and those solutions are $p_n + c$, where p_n is some particular solution that we know, and c is any constant.

Hence all the solutions of $\Delta c_n = n$ are

$$c_n = \frac{1}{2}n^{\underline{2}} + c,$$

all the solutions of $\Delta d_n = n^2$ are

$$d_n = \frac{1}{3}n^{\underline{3}} + \frac{1}{2}n^{\underline{2}} + c,$$

and all of the solutions of $\Delta e_n = n^3$ are

$$e_n = \frac{1}{4}n^{\underline{4}} + n^{\underline{3}} + \frac{1}{2}n^{\underline{2}} + c,$$

where c is any constant.

1.1 Exercises

Find all the solutions for the following difference equations. You may leave your solution in terms of falling factorials.

1. $\Delta a_n = n^4$

2. $\Delta b_n = 2n^2 - n + 4$

1.2 Solutions

1. We begin by noting that

$$n^{\underline{4}} = n(n-1)(n-2)(n-3) = n^4 - 6n^3 + 11n^2 - 6n$$

so

$$\begin{aligned} n^4 &= n^4 - 6n^3 + 11n^2 - 6n + 6(n^3 - 3n^2 + 2n) + 7(n^2 - n) + n \\ &= n^{\underline{4}} + 6n^{\underline{3}} + 7n^{\underline{2}} + n^{\underline{1}} \end{aligned}$$

Thus a solution of $\Delta a_n = n^4$ is

$$a_n = \frac{1}{5}n^{\underline{5}} + \frac{3}{2}n^{\underline{4}} + \frac{7}{3}n^{\underline{3}} + \frac{1}{2}n^{\underline{2}}.$$

Therefore, all the solutions are represented by

$$a_n + c = \frac{1}{5}n^{\underline{5}} + \frac{3}{2}n^{\underline{4}} + \frac{7}{3}n^{\underline{3}} + \frac{1}{2}n^{\underline{2}} + c$$

where c is any constant.

2. We write

$$\begin{aligned} 2n^2 - n + 4 &= 2(n^{\underline{2}} + n^{\underline{1}}) - n^{\underline{1}} + 4n^{\underline{0}} \\ &= 2n^{\underline{2}} - n^{\underline{1}} + 4n^{\underline{0}}. \end{aligned}$$

Thus a solution of $\Delta b_n = 2n^2 - n + 4$ is

$$b_n = \frac{2}{3}n^{\underline{3}} - \frac{1}{2}n^{\underline{2}} + 4n^{\underline{1}}.$$

Therefore, all the solutions are represented by

$$b_n + c = \frac{2}{3}n^{\underline{3}} - \frac{1}{2}n^{\underline{2}} + 4n^{\underline{1}} + c$$

where c is any constant.

2 The Difference Equation $\Delta a_n = a_n$

We now turn our attention to the last Take Home exercise. It asks us to find a solution of the following difference equation:

$$\Delta a_n = a_n.$$

We recall from the last lecture that

$$\Delta c \cdot b^n = c(b-1)b^n$$

for any number b and constant c. This is exactly what we want as long as $b-1=1$, that is, $b=2$. Thus $c2^n$ is a solution to our difference equation, for any constant c.

Are the solutions $c2^n$ the only solution to $\Delta a_n = a_n$? We analyze this question as follows. The difference equation $\Delta a_n = a_n$ means $a_{n+1} - a_n = a_n$, or

$$a_{n+1} - 2a_n = 0.$$

This is almost a difference equation. Can we somehow manipulate this equation to make it a difference equation? We can by dividing through the entire equation by 2^{n+1}:

$$\begin{aligned} a_{n+1} - 2a_n &= 0 \\ \frac{a_{n+1}}{2^{n+1}} - \frac{a_n}{2^n} &= 0 \qquad \text{(divide by } 2^{n+1}\text{)} \\ \Delta \frac{a_n}{2^n} &= 0 \end{aligned}$$

Look at that, a difference equation! By Lemma 1, $a_n/2^n = c$ for some number c. Therefore

$$a_n = c2^n.$$

This represents all the solutions of $\Delta a_n = a_n$. We state this as a theorem for convenience:

Theorem 2. *The only solutions of the difference equation $\Delta a_n = a_n$ are $a_n = c2^n$, where c is a constant.*

2.1 Exercises

1. Find all the solutions of the difference equation $\Delta a_n = \lambda a_n$, where λ is some real number. What happens to the solutions when $\lambda = -1$?

2. Find a solution to the difference equation $\Delta b_n = b_n + 1$.

2.2 Solutions

1. The solutions of the difference equation $\Delta a_n = \lambda a_n$ are

   (a) $a_n = c(1+\lambda)^n$ when $\lambda \neq -1$ (here c is an arbitrary constant);

   (b) the zero sequence $a_n = 0$ when $\lambda = -1$.

   Proof. $\Delta a_n = \lambda a_n$ is equivalent to the equation

   $$a_{n+1} - (1+\lambda)a_n = 0.$$

   If $\lambda \neq -1$, then $1+\lambda \neq 0$, and we can divide through our equation by $(1+\lambda)^{n+1}$, giving us the difference equation

   $$\Delta \frac{a_n}{(1+\lambda)^n} = 0.$$

   By Lemma 1, $\frac{a_n}{(1+\lambda)^n} = c$ for some constant c, and so

   $$a_n = c(1+\lambda)^n.$$

   If $\lambda = -1$, then we see that

   $$0 = a_{n+1} - (1+\lambda)a_n = a_{n+1}$$

   for all n. □

2. $b_n = -1$, for example

3 The Difference Equation $\Delta a_n = a_n + 1$

Let's take a look at the difference equation of exercise 2.1.2:

$$\Delta a_n = a_n + 1.$$

Were you able to come up with a solution? Our intuition tells us that the solutions of this equation should somehow be related to the solutions of $\Delta a_n = a_n$, namely $c2^n$. The next theorem tells us how they are related.

Theorem 3. *Let p_n be any solution of the difference equation $\Delta a_n = a_n + 1$. If b_n is any other solution, then $b_n = p_n + c2^n$ for some constant c.*

Proof. If p_n and b_n be are both solutions of $\Delta a_n = a_n + 1$, then by the linearity of Δ

$$\Delta(b_n - p_n) = \Delta b_n - \Delta p_n = b_n + 1 - (p_n + 1) = b_n - p_n.$$

Thus by Theorem 2, $b_n - p_n = c2^n$ for some constant c. Adding p_n to both sides of this equation gives

$$b_n = p_n + c2^n.$$

□

This theorem is really useful. It tells us that if we know just one solution of the difference equation $\Delta a_n = a_n + 1$, we actually know them all.

So how do we come up with a particular solution p_n of $\Delta a_n = a_n + 1$. The theory of how to do this in general is a little too advanced at this point. So what else can we do? We could try a sequence and hope we get lucky! Let's try a constant sequence $p_n = d$ for some constant d. We know $\Delta d = 0$, so the difference equation yields $0 = d + 1$, or $d = -1$. Wow! What luck! The constant sequence $p_n = -1$ solves the difference equation. By Theorem 3, we know that all of the solutions are of the form

$$a_n = p_n + c2^n = c2^n - 1.$$

3.1 Exercises

1. Find all the solutions of the difference equation $\Delta a_n = \lambda a_n + 1$, where λ is some real number. What happens to the solutions when $\lambda = 0$ or $\lambda = -1$?

2. Find all the solutions of the difference equation $\Delta b_n = b_n + 2$.

3.2 Solutions

1. The solutions to the difference equation $\Delta a_n = \lambda a_n + 1$ are

   (a) $a_n = c(1+\lambda)^n - 1$ when $\lambda \neq -1$ and $\lambda \neq 0$, where c is any constant;

   (b) $a_n = n + c$ when $\lambda = 0$, where c is any constant;

   (c) the constant sequence $a_n = 1$ when $\lambda = -1$.

   Proof. We already saw in Exercise 2.1.1 that the solutions of $\Delta d_n = \lambda d_n$ are

   (a) $c(1+\lambda)^n$ when $\lambda \neq -1$, for any constant c, and

   (b) the zero sequence 0 when $\lambda = -1$.

   Notice that if $\lambda \neq 0$, then $\frac{-1}{\lambda}$ is a particular solution of our difference equation. By a theorem similar to Theorem 3,

   $$c(1+\lambda)^n - \frac{1}{\lambda}$$

   represents all the solutions of $\Delta a_n = \lambda a_n + 1$ when $\lambda \neq -1$ and $\lambda \neq 0$.

   Of course, when $\lambda = 0$, our difference equation reduces to $\Delta a_n = 1$, which we can solve by the method of falling fractions, since $1 = n^{\underline{0}}$. Thus $n + c$ represents all solutions in this case.

   When $\lambda = -1$, we see that

   $$1 = a_{n+1} - (1+\lambda)a_n = a_{n+1}$$

   for all n. $\square$

2. The constant sequence -2 is a particular solution of $\Delta b_n = b_n + 2$ (try it!). Therefore, by a theorem similar to Theorem 3, $c2^n - 2$ represents all the solutions of $\Delta b_n = b_n + 2$, where c is any constant.

4 The Tower of Hanoi

We are now in position to solve an old and interesting mathematical puzzle – The Tower of Hanoi.

The Tower of Hanoi (sometimes referred to as the Tower of Brahma or the End of the World Puzzle) was invented by the French mathematician, Edouard Lucas, in 1883. He was inspired by a legend that tells of a Hindu temple where the pyramid puzzle might have been used for the mental discipline of young priests. Legend says that at the beginning of time the priests in the temple were given a stack of 64 gold disks, each one a little smaller than the one beneath it. Their assignment was to transfer the 64 disks from one of the three poles to another, with one important proviso – a large disk could never be placed on top of a smaller one. The priests worked very efficiently, day and night. When they finished their work, the myth said, the temple would crumble into dust and the world would vanish [1].

The mathematical puzzle is this – what is the least number of moves required to move n disks from the first pole to the last pole according to the rules given in the last paragraph? Let t_n represent this number. It would be great if we could come up with some recurrence relation for this sequence. Notice that if we have $n+1$ disks, we cannot move the bottom disk off the first pole until we have moved all the others off of it onto the other poles. Let's move the top n disks off to the second pole. That takes t_n moves. Then we can move the bottom disk to the third pole, requiring one move. We then move the n disks from the second pole onto the third pole, requiring another t_n moves. Thus

$$t_{n+1} = t_n + 1 + t_n = 2t_n + 1.$$

Subtracting t_n from both sides of this equation gives

$$\Delta t_n = t_n + 1$$

which is exactly the difference equation we just solved! Hence

$$t_n = c2^n - 1$$

for some constant c. We know that $t_1 = 1$ by trying it ourselves (duh!), and we use this equation to compute c:

$$1 = t_1 = c2^1 - 1 = 2c - 1$$

or $c = 1$. Therefore,

$$t_n = 2^n - 1.$$

Remember the legend said that the priests in the temple had 64 disks to work with. The number of moves required to move these disks from the first pole to the third pole is therefore

$$t_{64} = 18,446,744,073,709,551,615.$$

[1] http://www.lawrencehallofscience.org/Java/Tower/index.html

This is a huge number! If the priests worked day and night, making one move every second it would take slightly more than 580 billion years to accomplish the job! This is an enormous amount of time, considering that the generally accepted age of the Earth (and the rest of our solar system) is about 4.55 billion years [2].

[2] http://www.talkorigins.org/faqs/faq-age-of-earth.html