EXTERIOR BLOCKS AND REFLEXIVE NONCROSSING PARTITIONS

by

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GRADUATE COMMITTEE APPROVAL

of a thesis submitted by

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ABSTRACT

EXTERIOR BLOCKS AND REFLEXIVE NONCROSSING PARTITIONS

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Department of Mathematics

Master of Science

This thesis defines an *exterior block* of a noncrossing partition, then gives a formula for the number of noncrossing partitions of the set $\{1, 2, \ldots, n\}$ with k exterior blocks, which is

$$\frac{k}{n}\binom{2n-k-1}{k-1}.$$

Certain identities involving Catalan numbers are derived from this formula. A formula for the number of noncrossing partitions fixed by the reflection of the dihedral group is also derived, which is

$$\binom{n}{\lfloor n/2 \rfloor}$$

the nth central binomial coefficient.

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- my wife, Tiraje, for her love and support, which continues to be the greatest blessing of my life;
- God and His Son, who make every day a hopeful possibility.

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1 Introduction

In 1972 [13], Kreweras introduced mathematics to *noncrossing partitions*; that is, a partition π of the set $[n] := \{1, 2, \ldots, n\}$ such that whenever $1 \leq a < b < c < d \leq n$ and a and c are in the same block of π and b and d are in the same block of π, then actually a, b, c and d are all in the same block of π (this is the standard definition equivalent, but not equal, to Kreweras' original). The collection of noncrossing partitions of $[n]$ is denoted by NC_n. We typically write noncrossing partitions using a '/' to delimit the blocks of the partition and a ',' to delimit the elements within each block. For example, the partition $\pi = \{\{1, 4, 6\}, \{2, 3\}, \{5\}, \{7\}, \{8, 10\}, \{9\}, \{11, 12\}\} \in \mathrm{NC}_{12}$ is typically written $\pi = 1, 4, 6/2, 3/5/7/8, 10/9/11, 12$. Notice that we have written the blocks in ascending order of their least element. Noncrossing partitions can be conveniently visualized in their *linear* or *circular* representations. For the linear representation, we place n nodes $1, 2, \ldots, n$ on a line, and indicate that two elements are in the same block by drawing an arc in the upper-half plane connecting the two. For the circular representation, we place n nodes $1, 2, \ldots, n$ on a circle, and indicate that two elements are in the same block by drawing a line segment in the interior of the circle connecting the two. Figures 1 and 2 give the linear and circular representations, respectively, of $1, 4, 6/2, 3/5/7/8, 10/9/11, 12$. Throughout this paper we will make use of both representations.

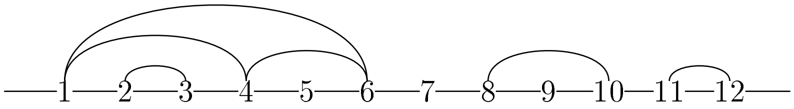

Figure 1: Linear representation of $1, 4, 6/2, 3/5/7/8, 10/9/11, 12$

Since Kreweras' paper, noncrossing partitions have been studied extensively in a variety of fields. For definitions not given in the discussion that follows, refer to [1] and [17]. It is well-known that NC_n, ordered by refinement (that is, $\pi \leq \sigma$ in NC_n if

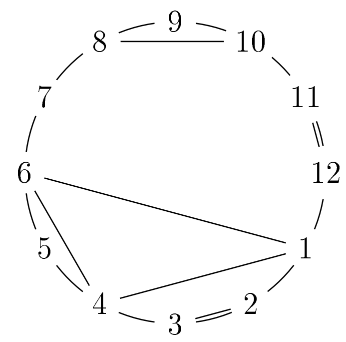

Figure 2: Circular representation of $1, 4, 6/2, 3/5/7/8, 10/9/11, 12$

for every block $B \in \pi$ there exists a block $C \in \sigma$ such that $B \subseteq C$), forms a graded lattice with rank function $\mathrm{rk}(\pi) = n - |\pi|$ (of course, $|\pi|$ denotes the number of blocks of π), and that $|\mathrm{NC}_n| = C_n$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth *Catalan number* [13]. The number of noncrossing partitions of $[n]$ of rank k, which is to ask for the number of noncrossing partitions with $n-k$ blocks, is $\frac{1}{n}\binom{n}{n-k}\binom{n}{n-k-1}$ [6]. NC_n possesses an infimum $\widehat{0} = 1/2/\cdots/n$ and a supremum $\widehat{1} = 1, 2, \ldots, n$ and the value of the Möbius function on the entire lattice is

$$\mu(\mathrm{NC}_n) = \mu_{\mathrm{NC}_n}(\widehat{0}, \widehat{1}) = (-1)^{n-1} C_{n-1}$$

[13] [8]. Many chain and multichain enumerations have been formulated, including its zeta polynomial

$$Z_{\mathrm{NC}_n}(m) = \frac{1}{n}\binom{mn}{n-1}$$

[13] [6] [7] [8]. It is known that NC_n is rank unimodal, rank symmetric, self-dual and admits a symmetric chain decomposition [15]. NC_n admits various R-labelings [4] [8], which have been used to characterize all parking functions, which in turn defines a local action of the symmetric group on NC_n [18]. The idea of a noncrossing partition has been generalized [14] and used to study classical reflection groups [3]. Noncrossing partitions are intimately connected with binary trees [12] and meanders [9] [10]. Recently, noncrossing partitions have been used to study stationary stochastic processes with freely independent increments [2].

So what possibly could there be left to study about noncrossing partitions? Consider the "easy" [15] problem of proving by induction that $|\mathrm{NC}_n| = C_n$. It seems to require the number of ways the singleton $\{n+1\}$ can be connected to a noncrossing partition $\pi \in \mathrm{NC}_n$ to get a noncrossing partition $\pi' \in \mathrm{NC}_{n+1}$. Figures 3, 4, 5, 6 and 7 illustrate an example of this problem for $n = 7$ and $\pi = 1,2/3/4,7/5,6$. We can certainly add the singleton $\{8\}$ to π to form $\pi' = \pi \cup \{\{8\}\} = 1,2/3/4,7/5,6/8$ as in Figure 3. We could also add it to the block $\{1,2\}$ of π to get $\pi' = (\pi \setminus \{\{1,2\}\}) \cup \{\{1,2,8\}\} = 1,2,8/3/4,7/5,6$ as in Figure 4; to the block $\{3\}$ to get $\pi' = (\pi \setminus \{\{3\}\}) \cup \{\{3,8\}\} = 1,2/3,8/4,7/5,6$ as in Figure 5; and to the block $\{4,7\}$ to get $\pi' = (\pi \setminus \{\{4,7\}\}) \cup \{\{4,7,8\}\} = 1,2/3/4,7,8/5,6$ as in Figure 6. If we add $\{8\}$ to the block $\{5,6\}$, we get $\pi' = (\pi \setminus \{\{5,6\}\}) \cup \{\{5,6,8\}\} = 1,2/3/4,7/5,6,8$, which is not a noncrossing partition (see Figure 7).

What is peculiar about the block $\{5,6\}$? Why does adding the singleton $\{8\}$ to it form a crossing partition? It is the fact that the block $\{5,6\}$ is "nested in" the block $\{4,7\}$. None of the other blocks of π are "nested in" blocks of π in this way. Notice that there are four distinct ways of adding $\{8\}$ to π to get π': one for each of the "unnested" blocks of π plus the case of adding $\{8\}$ as singleton to π.

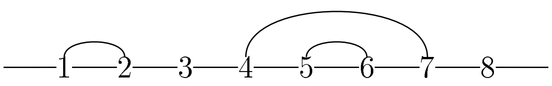

Figure 3: Linear representation of $1,2/3/4,7/5,6/8$

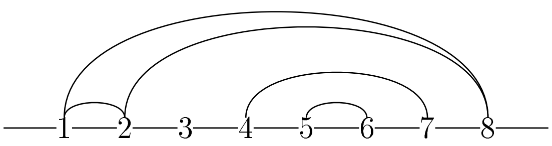

Figure 4: Linear representation of $1,2,8/3/4,7/5,6$

Consider a different problem. The dihedral group D_{2n} acts on the lattice NC_n

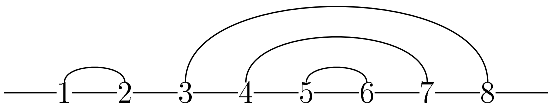

Figure 5: Linear representation of $1,2/3,8/4,7/5,6$

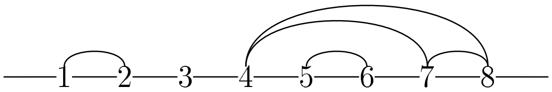

Figure 6: Linear representation of $1,2/3/4,7,8/5,6$

in a natural way (see Section 5 for details). We think of D_{2n} as being generated by the rotation element $r = (1,2,\ldots,n)$ and the reflection element $s = (2,n)(3,n-1)\cdots(\lceil n/2\rceil, n-\lceil n/2\rceil+2)$. What is this action like? Given any $\pi \in \mathrm{NC}_n$, what is the orbit of π under the action D_{2n}? How big is this orbit?

Let us begin with the easier task of describing the orbits of the subgroup $\langle s\rangle$ of D_{2n} (that is, the subgroup generated by the reflection s) acting on NC_n. The size of each orbit must divide the order of this subgroup, which is two [5]. If the size of a particular orbit is one, then the single noncrossing partition contained in this orbit is fixed by the action of s.

Consider such a noncrossing partition, $\pi = 1/2,4,7,9/3/5/6/8$ (see Figure 8; notice that π is symmetric about the dotted line). Notice that π can be constructed by reflecting the noncrossing partition $\beta = 1/2,4/3/5$ through the dotted line (see Figure 9) and then joining the block $\{2,4\}$ to its reflection $\{7,9\}$. Notice that in a similar way we could decide to connect the block $\{5\}$ to its reflection (see Figure 10),

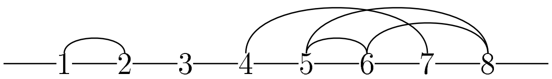

Figure 7: Linear representation of $1,2/3/4,7,8/5,6$

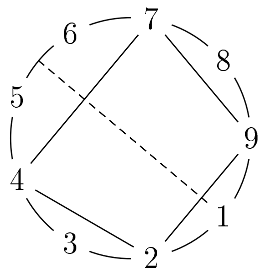

Figure 8: Circular representation of $1/2, 4, 7, 9/3/5/6/8$

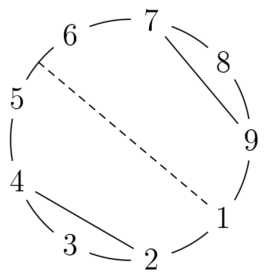

Figure 9: Circular representation of $1/2, 4/3/5/6/7, 9/8$

or connect both the blocks $\{2, 4\}$ and $\{5\}$ to their respective reflections (see Figure 11) to get a noncrossing partition fixed by the reflection s (the reflection of the block $\{1\}$ of π is simply $\{1\}$, so we get nothing new by connecting it to its reflection). However, if we try to connect the block $\{3\}$ to its reflection, we get a crossing partition (see Figure 12).

Why is the block $\{3\}$ different from the other blocks of β? Again, the block $\{3\}$ is "nested in" the block $\{2, 4\}$, while the other blocks of β are not "nested in" any other block of β. Notice that we constructed four distinct noncrossing partitions of $[9]$, all fixed by the reflection $s \in D_{18}$, from the noncrossing partition β: we had the choice to either connect or not connect the two "unnested" blocks of β with their reflections.

It seems that if we are to count the number of noncrossing partitions of $[n]$ fixed by the reflection $s \in D_{2n}$, or to build up NC_{n+1} from NC_n, or solve any other problem

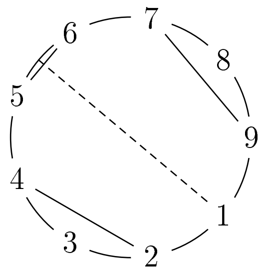

Figure 10: Circular representation of $1/2, 4/3/5, 6/7, 9/8$

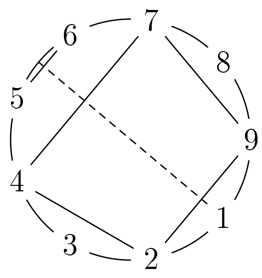

Figure 11: Circular representation of $1/2, 4, 7, 9/3/5, 6/8$

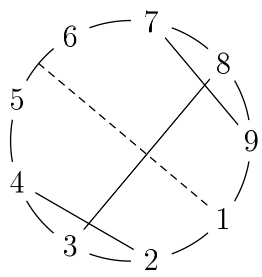

Figure 12: Circular representation of $1/2, 4/3, 8/5/6/7, 9$

dependent on the "nestedness" of the blocks of our noncrossing partitions, we will need to define and understand this concept of "nested" blocks.

2 Exterior Blocks

For ease of discussion we give a preliminary defintion. Given a block $B \in \pi$, we will denote the least and greatest elements of B by $\text{first}(B)$ and $\text{last}(B)$, respectively, and will call them the *first* and *last* elements of B, respectively.

Definition 2.1. Let $\pi \in \text{NC}_n$. A block $B \in \pi$ is an *interior block* of π if there exists a block $C \in \pi$ such that $\text{first}(C) < \text{first}(B) \leq \text{last}(B) < \text{last}(C)$. If B is not an interior block, then it is an *exterior block* of π.

Intuitively, given a noncrossing partition π of $[n]$, an interior block of π is one which is nested inside another block in the linear representation of π. An exterior block of π is one which is not nested in any other block. Consider Figure 1 which is the linear representation of $\pi = 1,4,6/2,3/5/7/8,10/9/11,12 \in \text{NC}_{12}$. It is easy to see that $\{2,3\}$, $\{5\}$ and $\{9\}$ are the interior blocks of π, while $\{1,4,6\}$, $\{7\}$, $\{8.10\}$ and $\{11,12\}$ are the exterior blocks of π.

We now present a few preliminary results concerning exterior blocks.

Propostion 2.1. *Let $\pi \in \text{NC}_n$. The blocks of π containing the elements 1 and n are always exterior blocks.*

Proof. Let A be the block of π containing the element 1, and B the block containing n. There is no block of π whose first element is less than $\text{first}(A) = 1$, thus, by definition, A is an exterior block of π. Similarly, B is an exterior block since there is no block of π whose last element is greater than $\text{last}(B) = n$. ☐

Corollary 2.1. *Every noncrossing partition of $[n]$ has at least one exterior block.*

Proof. This result follows immediately from Propostion 2.1 since every noncrossing partition of $[n]$ has a block that contains the element 1. ☐

Propostion 2.2. *Let $\pi \in \mathrm{NC}_n$. π has one exterior block if and only if the elements 1 and n are in the same block.*

Proof. ($\Rightarrow$) Suppose π has only one exterior block. If the elements 1 and n are not contained in the same block, then by Proposition 2.1 π has at least two exterior blocks, a contradiction. Therefore 1 and n must be in the same block.

($\Leftarrow$) Suppose 1 and n are in the same block B of π. Then every other block $A \in \pi$ is an interior block of π since

$$1 = \mathrm{first}(B) < \mathrm{first}(A) \leq \mathrm{last}(A) < \mathrm{last}(B) = n.$$

Therefore, B is the only exterior block of π. $\square$

3 The Function $\mathrm{e}(n, k)$

Let $\mathrm{E}_{n,k}$ be the subset of NC_n consisting of all noncrossing partitions of $[n]$ with k exterior blocks and define

$$\mathrm{e}(n, k) = |\mathrm{E}_{n,k}|$$

so that $\mathrm{e}(n, k)$ counts the number of noncrossing partitions of $[n]$ with k exterior blocks. What sort of function is $\mathrm{e}(n, k)$?

Propostion 3.1. $\mathrm{e}(n, k) = 0$ *whenever* $k = 0$ *or* $k > n$.

Proof. If $k = 0$, we are asking how many noncrossing partitions of $[n]$ have no exterior blocks. By Corollary 2.1 we know that there are no such noncrossing partitions. Thus $\mathrm{e}(n, 0) = 0$.

Since any partition of $[n]$ can have at most n blocks, it can have at most n exterior blocks. So if $k > n$, $\mathrm{e}(n, k) = 0$. $\square$

Propostion 3.2. $\mathrm{e}(n, k) > 0$ *whenever* $k \in [n]$.

Proof. Given $k \in [n]$, the noncrossing partition $1/2/\cdots/k-1/k, k+1, \ldots, n$ has k exterior blocks (see Figure 13). Therefore $\mathrm{e}(n, k) > 0$. $\square$

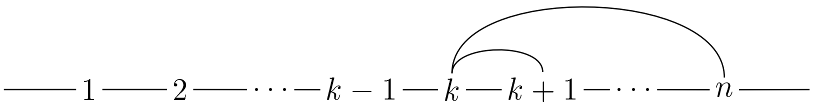

Figure 13: Linear representation of $1/2/\cdots/k-1/k,k+1,\ldots,n$

Propostion 3.3. $\mathrm{e}(n,n)=1$.

Proof. The only noncrossing partition of $[n]$ having n blocks is the infimum $\widehat{0} = 1/2/\cdots/n$ of the lattice NC_n. Notice that each singleton of $\widehat{0}$ is indeed an exterior block of $\widehat{0}$, so $\mathrm{e}(n,n)=1$. □

Propostion 3.4. $\mathrm{e}(n,n-1)=n-1$.

Proof. If a noncrossing partition π of $[n]$ has $n-1$ exterior blocks, then its blocks must all be singletons except for one block containing two *consecutive* elements; that is,

$$\pi = 1/\cdots/i-1/i,i+1/i+2/\cdots/n$$

for some $i \in [n-1]$ (see Figure 14). There are as many such noncrossing partitions as there are choices of i, which number is $n-1$. Therefore, $\mathrm{e}(n,n-1)=n-1$. □

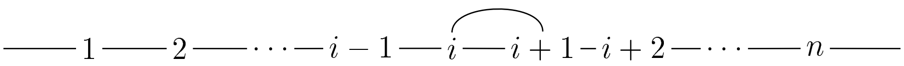

Figure 14: Linear representation of $1/\cdots/i-1/i,i+1/i+2/\cdots/n$

Theorem 3.1. $\mathrm{e}(n,1)=C_{n-1}$*, where* $C_n = \frac{1}{n+1}\binom{2n}{n}$ *is the nth* Catalan number.

Proof. We already know from Propostion 3.3 that $\mathrm{e}(1,1)=1=C_0$. Assume $n>1$. By Proposition 2.2, a noncrossing partition π of $[n]$ with one exterior block necessarily has 1 and n in the same block. Call this block B (see Figure 15, where $n=6$, $\pi = 1,4,6/2,3/5$ and $B=\{1,4,6\}$). The partition

$$\pi' = (\pi \setminus \{B\}) \cup \{B \setminus \{n\}\}$$

is then a noncrossing partition of $[n-1]$ (π' is simply π with the element n removed; see Figure 16). Define a map $\phi : \mathrm{E}_{n,1} \to \mathrm{NC}_{n-1}$ by the above operation $\pi \mapsto \pi'$. The map ϕ is clearly invertible, with inverse map ϕ^{-1} given by

$$\phi^{-1}(\sigma) = (\sigma \setminus \{A\}) \cup \{A \cup \{n\}\}$$

where $\sigma \in \mathrm{NC}_{n-1}$ and A is the block of σ containing the element 1 (see Figures 17 and 18, where $n = 6$, $\sigma = 1,2/3/4,5$ and $A = \{1,2\}$). Therefore ϕ is a bijection, proving

$$\mathrm{e}(n,1) = |\mathrm{E}_{n,1}| = |\mathrm{NC}_{n-1}| = C_{n-1}.$$

□

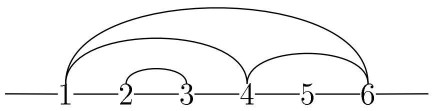

Figure 15: Linear representation of $\pi = 1,4,6/2,3/5 \in \mathrm{E}_{6,1}$

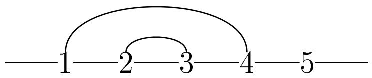

Figure 16: Linear representation of $\pi' = \phi(\pi) = 1,4/2,3/5 \in \mathrm{E}_{5,2}$

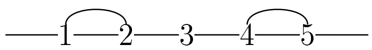

Figure 17: Linear representation of $\sigma = 1,2/3/4,5 \in \mathrm{E}_{5,3}$

So far we have only given the value of $\mathrm{e}(n,k)$ for particular values of k. We now prove a recurrence relation involving $\mathrm{e}(n,k)$.

Theorem 3.2. $\mathrm{e}(n,k) = \mathrm{e}(n-1,k-1) + \mathrm{e}(n,k+1)$ *for* $n \geq 2$ *and* $k \geq 1$.

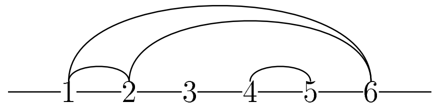

Figure 18: Linear representation of $\phi^{-1}(\sigma) = 1, 2, 6/3/4, 5 \in \mathrm{E}_{6,1}$

Proof. Clearly $\mathrm{e}(n-1, k-1)$ counts the number of noncrossing partitions π of $\mathrm{E}_{n,k}$ having the singleton $\{n\}$ as a block since $\pi \setminus \{n\} \in \mathrm{E}_{n-1,k-1}$. Thus we want to show that $\mathrm{e}(n, k+1)$ counts the number of noncrossing partitions of $\mathrm{E}_{n,k}$ that do not have $\{n\}$ as a block. Let $\mathrm{E}'_{n,k}$ be that set.

If $k \in [n-1]$ then by the example in Proposition 3.2 there exists a noncrossing partition with k exterior blocks whose block containing n is not a singleton. Thus if $\mathrm{E}'_{n,k}$ is empty, then necessarily $k \geq n$. But then $\mathrm{e}(n, k+1) = 0$ by Propostion 3.1 and we are done.

If $\mathrm{E}'_{n,k}$ is not empty, then for any $\pi \in \mathrm{E}'_{n,k}$, let B be the block of π containing n and let

$$\pi' = (\pi \setminus \{B\}) \cup \{B \setminus \{n\}, \{n\}\}$$

(see Figures 19 and 20, where $n = 6$, $k = 2$, $\pi = 1, 2/3, 4, 6/5$ and $B = \{3, 4, 6\}$). Now π' is a noncrossing partition of $[n]$ with more than k exterior blocks. Let C be the block of π' just to the right of $B \setminus \{n\}$ in the linear representation of π'; that is, $\mathrm{last}(B \setminus \{n\}) + 1 = \mathrm{first}(C)$ ($B \setminus \{6\} = \{3, 4\}$ and $C = \{5\}$ in Figure 20). Let

$$\pi'' = (\pi \setminus \{C, \{n\}\}) \cup \{C \cup \{n\}\}$$

(see Figure 21). Now $\pi'' \in \mathrm{E}_{n,k+1}$. Define a map $\psi : \mathrm{E}'_{n,k} \to \mathrm{E}_{n,k+1}$ by the above operation $\pi \mapsto \pi''$. The map ψ is clearly invertible with inverse map ψ^{-1} given by

$$\psi^{-1}(\sigma) = (\sigma \setminus \{A\}) \cup \{D \cup \{n\}, A \setminus \{n\}\}$$

where $\sigma \in \mathrm{E}_{n,k+1}$ and A is the block of σ containing n and D is the block of σ just to the left of A in the linear representation of σ; that is, $\mathrm{last}(D) + 1 = \mathrm{first}(A)$ (see

Figures 22 and 23, where $n = 6$, $k = 2$, $\sigma = 1, 2/3/4, 5, 6$, $A = \{4, 5, 6\}$ and $D = \{3\}$). Therefore, ψ is a bijection and

$$\left|\mathrm{E}'_{n,k}\right| = \left|\mathrm{E}_{n,k+1}\right| = \mathrm{e}(n, k+1).$$

We have proven the desired recurrence. $\square$

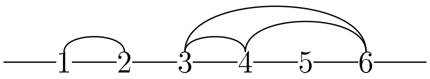

Figure 19: Linear representation of $\pi = 1, 2/3, 4, 6/5 \in \mathrm{E}'_{6,2}$

Figure 20: Linear representation of $\pi' = 1, 2/3, 4/5/6 \in \mathrm{E}_{6,4}$

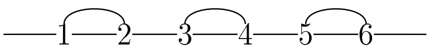

Figure 21: Linear representation of $\pi'' = \psi(\pi) = 1, 2/3, 4/5, 6 \in \mathrm{E}_{6,3}$

This recurrence relations allows us to write out a table of values for $\mathrm{e}(n, k)$ (see Figure 24). Notice that the values of the first two columns of this table come from Proposition 3.1 and Theorem 3.1, while the rest of the values come from the recurrence relation written as $\mathrm{e}(n, k+1) = \mathrm{e}(n, k) - \mathrm{e}(n-1, k-1)$.

Catalan numbers abound in this table. Notice that the second and third columns (corresponding to $k = 1$ and $k = 2$) contain Catalan numbers. The first column is, of course, given to us by Theorem 3.1. When $k = 2$ and $n \geq 2$, the recurrence relation plus Proposition 3.1 shows us that

$$\mathrm{e}(n, 2) = \mathrm{e}(n, 1) - \mathrm{e}(n-1, 0) = C_{n-1} - 0 = C_{n-1}.$$

Notice that the nth row adds up to C_n. This is clear since the sets

$$\mathrm{E}_{n,1}, \mathrm{E}_{n,2}, \ldots, \mathrm{E}_{n,n}$$

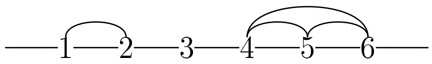

Figure 22: Linear representation of $\sigma = 1,2/3/4,5,6 \in \mathrm{E}_{6,3}$

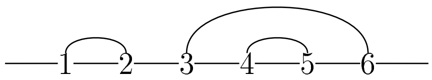

Figure 23: Linear representation of $\psi^{-1}(\sigma) = 1,2/3,6/4,5 \in \mathrm{E}'_{6,2}$

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	Total
1	0	1	0	0	0	0	0	0	0	0	0	1
2	0	1	1	0	0	0	0	0	0	0	0	2
3	0	2	2	1	0	0	0	0	0	0	0	5
4	0	5	5	3	1	0	0	0	0	0	0	14
5	0	14	14	9	4	1	0	0	0	0	0	42
6	0	42	42	28	14	5	1	0	0	0	0	132
7	0	132	132	90	48	20	6	1	0	0	0	429
8	0	429	429	297	165	75	27	7	1	0	0	1430
9	0	1430	1430	1001	572	275	110	35	8	1	0	4862
10	0	4862	4862	3432	2002	1001	429	154	44	9	1	16796

Figure 24: Table of values of $\mathrm{e}(n,k)$

partition NC_n; that is, the sets are pairwise disjoint and $\mathrm{NC}_n = \cup_{k=1}^{n} \mathrm{E}_{n,k}$. This fact gives

$$C_n = |\mathrm{NC}_n| = |\cup_{k=1}^{n} \mathrm{E}_{n,k}| = \Sigma_{k=1}^{n} |\mathrm{E}_{n,k}| = \Sigma_{k=1}^{n} \mathrm{e}(n,k). \tag{3.1}$$

Also notice the strong resemblance of this table with the various formulations of *Catalan's triangle* (cf. [11], also sequences A053121, A008315, etc. in [16]). Figure 25 is a typical Catalan triangle. It is also called a Pascal semi-triangle since if $w(n,k)$ represents the value in the nth row and kth column of this table, then for $n \geq 1$ and $k \geq 1$, $w(n,k)$ satisfies the recurrence relation

$$w(n,k) = w(n-1,k-1) + w(n-1,k+1).$$

Notice that the diagonals $w(2n,0), w(2n-1,1), \ldots, w(n,n)$ of this triangle are the rows $\mathrm{e}(n+1,1), \mathrm{e}(n+1,2), \ldots, \mathrm{e}(n+1,n+1)$ in Figure 24.

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	Total
0	1	0	0	0	0	0	0	0	0	0	0	1
1	0	1	0	0	0	0	0	0	0	0	0	1
2	1	0	1	0	0	0	0	0	0	0	0	2
3	0	2	0	1	0	0	0	0	0	0	0	3
4	2	0	3	0	1	0	0	0	0	0	0	6
5	0	5	0	4	0	1	0	0	0	0	0	10
6	5	0	9	0	5	0	1	0	0	0	0	20
7	0	14	0	14	0	6	0	1	0	0	0	35
8	14	0	28	0	20	0	7	0	1	0	0	70
9	0	42	0	48	0	27	0	8	0	1	0	126
10	42	0	90	0	75	0	35	0	9	0	1	252

Figure 25: A Catalan Triangle

We are now ready to give a closed formula for $\mathrm{e}(n,k)$.

Theorem 3.3. $\mathrm{e}(n,k) = \frac{k}{n}\binom{2n-k-1}{n-1}$.

Proof. Let $f(n,k) = \frac{k}{n}\binom{2n-k-1}{n-1}$. Notice that if $k=0$ then

$$f(n,0) = \frac{0}{n}\binom{2n-1}{n-1} = 0 = \mathrm{e}(n,0)$$

and if $k = 1$ then

$$f(n,1) = \frac{1}{n}\binom{2n-2}{n-1} = C_{n-1} = \mathrm{e}(n,1)$$

and if $k > n$ then $2n - k - 1 < 2n - n - 1 = n - 1$ so that

$$f(n,k) = \frac{k}{n}\binom{2n-k-1}{n-1} = \frac{k}{n} \cdot 0 = 0 = \mathrm{e}(n,k)$$

if we follow the convention that $\binom{a}{b} = 0$ whenever $b > a$. Hence $f(n,k)$ satisfies the initial conditions of Proposition 3.1 and Theorem 3.1. It remains to show that this formula satisfies the recurrence relation

$$f(n,k) = f(n-1,k-1) + f(n,k+1);$$

that is,

$$\begin{aligned} \frac{k}{n}\binom{2n-k-1}{n-1} &= \frac{k-1}{n-1}\binom{2(n-1)-(k-1)-1}{(n-1)-1} + \frac{k+1}{n}\binom{2n-(k+1)-1}{n-1} \\ &= \frac{k-1}{n-1}\binom{2n-k-2}{n-2} + \frac{k+1}{n}\binom{2n-k-2}{n-1} \end{aligned}$$

for $n \geq 2$ and $k \geq 1$. Here we go:

$$\begin{aligned}
f(n-1,k-1)+f(n,k+1) &= \frac{k-1}{n-1}\binom{2n-k-2}{n-2} + \frac{k+1}{n}\binom{2n-k-2}{n-1} \\
&= \frac{k-1}{n-1}\binom{2n-k-2}{n-2} + \frac{k+1}{n}\binom{2n-k-2}{n-1} \\
&= \frac{k-1}{n-1}\cdot\frac{(2n-k-2)!}{(n-2)!(n-k)!} \\
&\quad + \frac{k+1}{n}\cdot\frac{(2n-k-2)!}{(n-1)!(n-k-1)!} \\
&= \frac{n(k-1)(2n-k-2)!}{n!(n-k)!} \\
&\quad + \frac{(n-k)(k+1)(2n-k-2)!}{n!(n-k)!} \\
&= \frac{(nk-n)(2n-k-2)!}{n!(n-k)!} \\
&\quad + \frac{(nk+n-k^2-k)(2n-k-2)!}{n!(n-k)!} \\
&= \frac{(2nk-k^2-k)(2n-k-2)!}{n!(n-k)!} \\
&= \frac{k}{n}\cdot\frac{(2n-k-1)(2n-k-2)!}{(n-1)!(n-k)!} \\
&= \frac{k}{n}\cdot\frac{(2n-k-1)!}{(n-1)!(n-k)!} \\
&= \frac{k}{n}\binom{2n-k-1}{n-1} = f(n,k).
\end{aligned}$$

Therefore, $\mathrm{e}(n,k) = f(n,k) = \frac{k}{n}\binom{2n-k-1}{n-1}$. □

4 Catalan Identities

Using the formulation $\mathrm{e}(n,k) = \frac{k}{n}\binom{2n-k-1}{n-1}$ of Theorem 3.3, we can derive some identities involving Catalan numbers. The first comes by replacing $\mathrm{e}(n,k)$ in Equation 3.1 by this formula:

$$C_n = \sum_{k=1}^{n} \mathrm{e}(n,k) = \sum_{k=1}^{n} \frac{k}{n}\binom{2n-k-1}{n-1}.$$

The second identity comes in response to the question posed at the beginning of this paper regarding a proof by induction of the fact $|\mathrm{NC}_n| = C_n$ (see Section 1).

There we asked the number of ways the element $n+1$ can be added to a noncrossing partition π of $[n]$ to get a noncrossing partition π' of $[n+1]$, and concluded that if π has k exterior blocks, then there are $k+1$ ways to form the new noncrossing partition π'. Since there are $\mathrm{e}(n,k)$ noncrossing partitions of $[n]$ with k exterior blocks, there are a total of $(k+1)\mathrm{e}(n,k)$ noncrossing partitions of $[n+1]$ gotten in this way. Summing these formulae over the possible number of exterior blocks gives

$$C_{n+1} = |\mathrm{NC}_{n+1}| = \sum_{k=1}^{n}(k+1)\mathrm{e}(n,k) = \sum_{k=1}^{n}\frac{k(k+1)}{n}\binom{2n-k-1}{n-1}.$$

5 The Action of the Dihedral Group D_{2n} on the Lattice NC_n

The dihedral group D_{2n} acts in a natural way on the lattice NC_n. We typically present the dihedral group as $D_{2n} = \langle r, s \mid r^n, s^2, rsrs\rangle$, and call the generator r the *rotation* in D_{2n} and the generator s the *reflection* in D_{2n}. Considered as a subgroup of the symmetric group on n letters, the generators are typically written $r = (1,2,\ldots,n)$ and $s = (2,n)(3,n-1)\cdots(\lceil n/2\rceil, n-\lceil n/2\rceil+2)$ [5].

A *group action* of a group G on a set A is a map $\cdot : G\times A \to A$ (where $\cdot(g,a)$ is typically written $g\cdot a$, or even ga) such that for all $g_1, g_2 \in G$ and $a \in A$ the following two properties hold:

1. $g_1\cdot(g_2\cdot a) = (g_1g_2)\cdot a$, and

2. $1\cdot a = a$ (1 is the identity element of G).

It follows from Property (1) that the action of any group G on a set A is determined by the action of the generators of G on the set A [5].

The group D_{2n} acts on the lattice NC_n by simply permuting the elements $1,2,\ldots,n$ of the blocks of any noncrossing partition $\pi \in \mathrm{NC}_n$. For instance, if $n=5$ and $\pi = 1,3/2/4,5 \in \mathrm{NC}_5$, then $r = (1,2,3,4,5)$, $s = (2,5)(3,4)$ and

$$r\cdot\pi = (1,2,3,4,5)\cdot 1,3/2/4,5 = 1,5/2,4/3 \in \mathrm{NC}_5$$

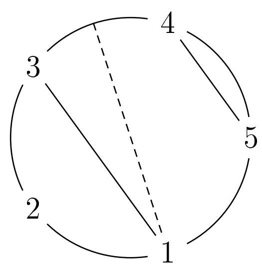

Figure 26: Circular representation of $\pi = 1,3/2/4,5$

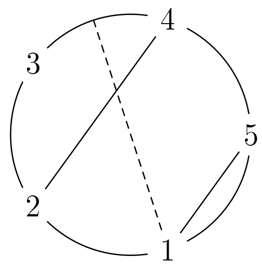

Figure 27: Circular representation of $r \cdot \pi = 1,5/2,4/3$

and

$$s \cdot \pi = (2,5)(3,4) \cdot 1,3/2/4,5 = 1,4/2,3/5 \in \mathrm{NC}_5.$$

Figures 26, 27 and 28 illustrate these examples (the dotted line represents the axis in which the generator s reflects π). It is easily seen that the actions of r and s on π do not change the block structure (that is, the number and size of each block, the block adjacencies, etc.) of π; they simply rotate or reflect it. This fact is true for the actions of r and s on any noncrossing partition of $[n]$.

An interesting property of the action of D_{2n} on NC_n is that it is rank- and order-preserving. To say that the action is *rank-preserving* means that for any $\pi \in \mathrm{NC}_n$ and $d \in D_{2n}$, $\mathrm{rk}(\pi) = \mathrm{rk}(d \cdot \pi)$. To say that the action is *order-preserving* means that if $\pi \leq \sigma$ for any $\sigma \in \mathrm{NC}_n$, then $d \cdot \pi \leq d \cdot \sigma$. This follows immediately from the fact that the action is does not change the block structure of a noncrossing partition.

Because the action is rank-preserving and the height of NC_n is $n-1$ (that is, there

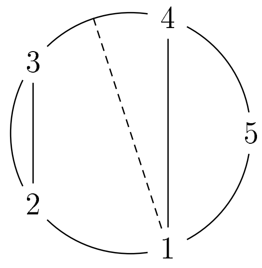

Figure 28: Circular representation of $s \cdot \pi = 1, 4/2, 3/5$

are n distinct ranks), when $n > 1$ this action is not transitive; that is, there is more than one orbit of this action. We are thus led to ask about the number and size of the orbits of this action.

6 Reflexive Noncrossing Partitions

In this paper we will only consider the orbits of NC_n under the action of the subgroup $\langle s \rangle$ of D_{2n}. As was mentioned in Section 1, the orbits under this action must have sizes that divide the order of this subgroup, which is two. Hence the orbits have either size one or two. If an orbit has size one, then the single noncrossing partition contained in this orbit is fixed by the action of s. Thus if we can count the number of noncrossing partitions of $[n]$ fixed by s, then the rest of the noncrossing partitions must be paired up into orbits of size two. We will call any noncrossing partition fixed by the reflection s a *reflexive* noncrossing partition. The number of reflexive noncrossing partitions of $[n]$ will be denoted by $s(n)$.

Theorem 6.1. *The number of reflexive noncrossing partitions of $[n]$ is the nth* central binomial coefficient $\binom{n}{\lfloor n/2 \rfloor}$*; that is*

$$s(n) = \binom{n}{\lfloor n/2 \rfloor}.$$

Corollary 6.1. *There are $\frac{1}{2}\left[C_n + \binom{n}{\lfloor n/2 \rfloor}\right]$ orbits of $\langle s \rangle$ acting on NC_n: $\binom{n}{\lfloor n/2 \rfloor}$ orbits of size one and $\frac{1}{2}\left[C_n - \binom{n}{\lfloor n/2 \rfloor}\right]$ orbits of size two.*

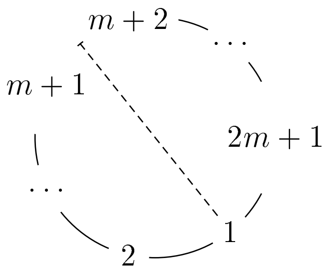

Figure 29: The circular representation of any noncrossing partition of $[n]$, where $n = 2m+1 > 1$, has m places $2, 3, \ldots, m+1$ on one side of the axis of reflection

The proof of Theorem 6.1 will be divided into two cases: one in case n is odd, and the other in case n is even.

Proof of Theorem 6.1 in case $n = 2m+1$ is odd. If $n = 1$, then NC_1 contains only one noncrossing partition, namely 1, which is clearly reflexive. Since

$$\binom{1}{\lfloor 1/2 \rfloor} = 1 = s(1),$$

the formula holds for $n = 1$.

Assume $n > 1$. We will show how to construct the reflexive noncrossing partitions of $[n]$. Since $n > 1$, it follows that $m \geq 1$, and in the circular representation of any noncrossing partition of $[n]$ there are m places labelled $2, 3, \ldots, m+1$ on one side of the axis of reflection (see Figure 29). In these m places we can put a noncrossing partition β of $[2, m+1] = \{2, 3, \ldots, m+1\}$ with k exterior blocks (see Figure 30). Now, we add the singleton $\{1\}$ to β by simply adding the block or by adding it to any one of the exterior blocks of β to get a new noncrossing partition β' of $[m+1]$ with $k' \leq k+1$ exterior blocks (see Figure 31). We then reflect β' in the axis of reflection to get a reflexive noncrossing partition $\pi \in \mathrm{NC}_n$ (see Figure 32). We can form more reflexive noncrossing partitions from β' by choosing to connect the exterior blocks of β' not containing 1 with their respective reflections (see Figure 33).

There are $\mathrm{e}(m, k)$ ways to choose β. If we do not connect $\{1\}$ to any of the exterior blocks of β, then β' has $k' = k+1$ exterior blocks, so that there are $2^{k'-1} = 2^k$ ways

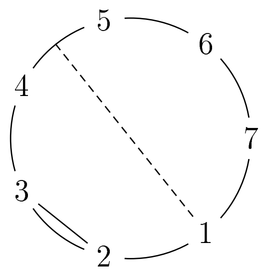

Figure 30: Circular representation with $n = 7$, $m = 3$ and $\beta = 2, 3/4$

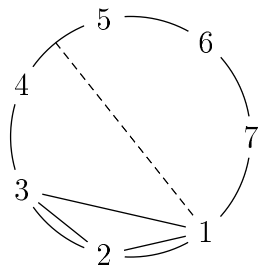

Figure 31: Circular representation with $n = 7$, $m = 3$, and $\beta' = 1, 2, 3/4$

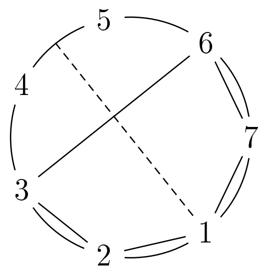

Figure 32: Circular representation of the reflection of $\beta' = 1, 2, 3/4$ through the axis of reflection, giving $\pi = 1, 2, 3, 6, 7/4/5$

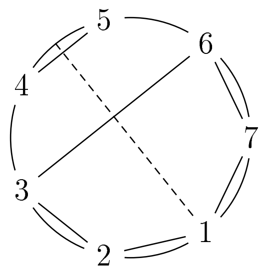

Figure 33: Circular representation of the reflection of $\beta' = 1,2,3/4$ through the axis of reflection and connecting the block $\{4\}$ with its reflection $\{5\}$, giving $\pi = 1,2,3,6,7/4,5$

to construct a reflexive noncrossing partition of $[n]$ from β'. Hence in this way we can construct $2^k\mathrm{e}(m,k)$ reflexive noncrossing partitions of $[n]$.

If we connect $\{1\}$ to the first exterior block of β, then β' has $k' = k$ exterior blocks, so that there are $2^{k'-1} = 2^{k-1}$ ways to construct a reflexive noncrossing partition of $[n]$ from β'. Hence in this way we can construct $2^{k-1}\mathrm{e}(m,k)$ reflexive noncrossing partitions of $[n]$.

In general, if we connect $\{1\}$ to the jth exterior block of β, β' has $k' = k - j + 1$ exterior blocks, so that there are $2^{k-j+1-1} = 2^{k-j}$ ways to construct a reflexive noncrossing partition of $[n]$ from β'. Hence in this way we can construct $2^{k-j}\mathrm{e}(m,k)$ reflexive noncrossing partitions of $[n]$. Summing the expression $2^{k-j}\mathrm{e}(m,k)$ from $j = 0$ to k ($j = 0$ corresponds to the case where $\{1\}$ is not connected to any of the exterior blocks of β') gives

$$\sum_{j=0}^{k} 2^{k-j}\mathrm{e}(m,k) = (2^{k+1} - 1)\mathrm{e}(m,k)$$

ways to construct reflexive noncrossing partitions of $[n]$ from noncrossing partitions of m having k exterior blocks. If we now sum this expression over the possible number of exterior blocks we get a formula for $s(n)$ when $n > 1$ is odd:

$$\begin{aligned}
s(n) &= \sum_{k=1}^{m}(2^{k+1}-1)\mathrm{e}(m,k) \\
&= \sum_{k=1}^{m} 2^{k+1}\mathrm{e}(m,k) - \sum_{k=1}^{m}\mathrm{e}(m,k) \\
&= \sum_{k=1}^{m} \frac{2^{k+1}k}{m}\binom{2m-k-1}{m-1} - C_m \\
&= 4(2m-1)C_{m-1} - C_m \\
&= \binom{2m+1}{m} = \binom{n}{\lfloor n/2 \rfloor}
\end{aligned}$$

□

Proof of Theorem 6.1 in case $n = 2m$ is even. If $n = 2$, then there are two noncrossing partitions of [2], namely $1, 2$ and $1/2$, both of which are clearly reflexive. Since

$$\binom{2}{\lfloor 2/2 \rfloor} = 2 = s(2),$$

the formula holds for $n = 2$.

Assume $n > 2$. Then $m \geq 1$, and in the circular representation of any noncrossing partition of $[n]$ there will be $m-1$ places labelled $2, 3, \ldots, m$ on one side of the axis of reflection (see Figure 34). If we ignore the place $m+1$, then we are in the case studied above, namely counting the number of reflexive noncrossing partitions of $[n-1]$, of which there are

$$s(n-1) = \binom{n-1}{\lfloor (n-1)/2 \rfloor} = \binom{2m-1}{m-1}. \tag{6.1}$$

Notice that $\{m+1\}$ is a singleton in each of noncrossing partitions counted here.

If we ignore the place 1, then we are again in the same case. As with the case of ignoring the place $m+1$, all the noncrossing partitions that we will count now will have $\{1\}$ as a singleton block. Notice that we have already counted all those reflexive noncrossing partitions that have both $\{1\}$ and $\{m+1\}$ as singleton blocks. So to avoid double-counting, we must avoid constructing them. If we put a noncrossing partition

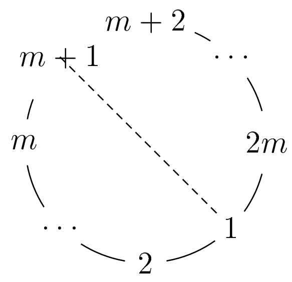

Figure 34: The circular representation of any noncrossing partition of $[n]$, where $n = 2m > 2$, has $m-1$ places $2, 3, \ldots, m$ on one side of the axis of reflection

having k exterior blocks into the $m-1$ places $2, 3, \ldots, m$, then by the procedure described above we can construct

$$\sum_{j=0}^{k} 2^{k-j}\mathrm{e}(m-1,k) = (2^{k+1}-1)\mathrm{e}(m-1,k)$$

reflexive noncrossing partitions from it. Remember that the index j represents the position of the exterior block to which we are connecting the block $\{m+1\}$, and that $j = 0$ represents no connection to the block $\{m+1\}$. This is the case we want to eliminate to avoid double-counting, so we simply subtract $2^k\mathrm{e}(m-1,k)$ from the sum above (which amounts to starting the sum at $j = 1$) to get

$$\sum_{j=1}^{k} 2^{k-j}\mathrm{e}(m-1,k) = (2^{k}-1)\mathrm{e}(m-1,k)$$

more reflexive noncrossing partitions of $[n]$. Summing this number over the possible number of exterior blocks gives

$$\begin{aligned}
\sum_{k=1}^{m-1}(2^k-1)\mathrm{e}(m-1,k) &= \sum_{k=1}^{m-1}2^k\mathrm{e}(m-1,k)-\sum_{k=1}^{m-1}\mathrm{e}(m-1,k)\\
&= \sum_{k=1}^{m-1}2^k\mathrm{e}(m-1,k)-C_{m-1}\\
&= \frac{1}{2}\left(\sum_{k=1}^{m-1}2^{k+1}\mathrm{e}(m-1,k)-C_{m-1}\right)-\frac{1}{2}C_{m-1} \qquad (6.2)\\
&= \frac{1}{2}s(n-1)-\frac{1}{2}C_{m-1}\\
&= \frac{1}{2}\binom{n-1}{\lfloor (n-1)/2\rfloor}-\frac{1}{2}C_{m-1}\\
&= \frac{1}{2}\binom{2m-1}{m-1}-\frac{1}{2}C_{m-1}
\end{aligned}$$

What do we have left to count? We have not considered those reflexive noncrossing partitions which have 1 and $m+1$ in the same block. How do we count these? If we put a noncrossing partition with k exterior blocks into the $m-1$ places (see Figure 35), we have a choice to connect or not connect each of the exterior blocks one at a time to the block $\{1, m+1\}$ (see Figure 36), and then reflect that in the axis of reflection (see Figure 37). We connect the exterior blocks one at a time to avoid double-counting. This is completely analogous to the situation discussed earlier of building up the noncrossing partitions of $[m]$ from those of $[m-1]$. Thus there are

$$C_m \qquad (6.3)$$

such reflexive noncrossing partitions.

Now what is left to count? The only reflexive noncrossing partitions we have not yet counted are those which do not have 1 and $m+1$ in singleton blocks or in the same block. Again, we begin by putting a noncrossing partition β with k exterior blocks into the $m-1$ places. If we add 1 to the first exterior block of β, then we have the choice of adding $m+1$ to any of the other exterior blocks of β. If we add it to the second, then we now have a noncrossing partition β' of $[m+1]$ with two exterior

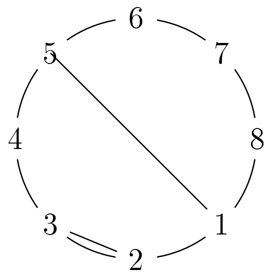

Figure 35: Circular representation with $n = 8$, $m = 4$, $k = 2$ and noncrossing partition $\beta = 2, 3/4$

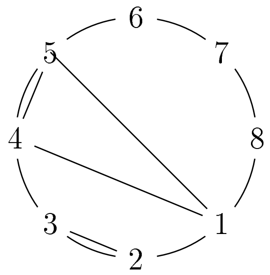

Figure 36: Circular representation adding the block $\{4\}$ of β to the block $\{1, 5\}$ to get $\beta' = 1, 4, 5/2, 3$

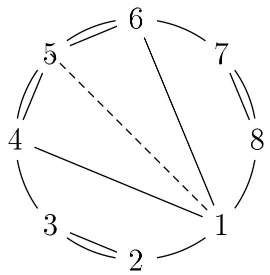

Figure 37: Circular representation reflecting β' in the axis of reflection to get $\pi = 1, 4, 5, 6/2, 3/7, 8$

blocks. We then reflect β' in the axis of reflection to get a reflexive noncrossing partition π of $[n]$. Connecting the exterior blocks of β' to there reflections gives the same noncrossing partition π since both contain either 1 or $m+1$. Thus there are as many of these reflexive noncrossing partitions as there are choices of β, which is $\mathrm{e}(m-1,k)$.

If we instead add $m+1$ to the third exterior block of β to get β' and then reflect β' in the axis of reflection, we get a reflexive noncrossing partition π of $[n]$. Notice that β' has three exterior blocks, and if we decide to connect the second exterior block of β' with its reflection we get another reflexive noncrossing partition π of $[n]$. Thus there are 2 reflexive noncrossing partitions that can be constructed from β in this way. There are $\mathrm{e}(m-1,k)$ choices for β, so there are $2\mathrm{e}(m-1,k)$ reflexive noncrossing partitions constructed in this way.

In general, if we put a noncrossing partition with k exterior blocks into the $m-1$ places, and if we add 1 to the first exterior block and $m+1$ to the ith exterior block, where $i > 1$, we can construct 2^{i-2} reflexive noncrossing partitions of $[n]$ from it. There are $\mathrm{e}(m-1,k)$ choices for the original partition, so there are $2^{i-2}\mathrm{e}(m-1,k)$ reflexive noncrossing partitions that can be constructed in this way. Adding all these possibilities together, we get

$$\sum_{i=2}^{k} 2^{i-2}\mathrm{e}(m-1,k) = \sum_{i=0}^{k-2} 2^{i}\mathrm{e}(m-1,k)$$

reflexive noncrossing partitions of $[n]$.

If we instead add 1 to the jth exterior block of β, we can only add $m+1$ to the ith exterior block of β if $i > j$, else we will have a crossing partition if $i < j$, or a noncrossing partition we have already counted if $i = j$ (since then 1 and $m+1$ would be in the same block). As above, there will be 2^{i-j-1} ways to get a reflexive a noncrossing partition of $[n]$ from β, and there are $\mathrm{e}(m-1,k)$ choices for β, so there are $2^{i-j-1}\mathrm{e}(m-1,k)$ reflexive noncrossing partitions of $[n]$ constructed in this way.

Adding all these possibilities together, we get

$$\sum_{i=j+1}^{k} 2^{i-j-1}\mathrm{e}(m-1,k) = \sum_{i=0}^{k-j-1} 2^{i}\mathrm{e}(m-1,k)$$

reflexive noncrossing partitions of $[n]$. So the number of all of the reflexive noncrossing partitions of $[n]$ that can be constructed in this way from noncrossing partitions with k exterior blocks put into the $m-1$ places is

$$\begin{aligned}
\sum_{j=1}^{k-1}\sum_{i=0}^{k-j-1} 2^{i}\mathrm{e}(m-1,k) &= \sum_{j=0}^{k-2}\sum_{i=0}^{k-j-2} 2^{i}\mathrm{e}(m-1,k)\\
&= \sum_{j=0}^{k-2}\sum_{i=0}^{j} 2^{i}\mathrm{e}(m-1,k)\\
&= \sum_{j=0}^{k-2}(2^{j+1}-1)\mathrm{e}(m-1,k)\\
&= \left[\sum_{j=0}^{k-2} 2^{j+1} - \sum_{j=0}^{k-2} 1\right]\mathrm{e}(m-1,k)\\
&= \left[2^{k}-2-(k-1)\right]\mathrm{e}(m-1,k)\\
&= (2^{k}-k-1)\mathrm{e}(m-1,k)
\end{aligned}$$

If we now sum this expression over the possible number of exterior blocks we get

$$\begin{aligned}
\sum_{k=1}^{m-1}\sum_{j=0}^{k-2}\sum_{i=0}^{j} 2^{i}\mathrm{e}(m-1,k) &= \sum_{k=1}^{m-1}(2^{k}-k-1)\mathrm{e}(m-1,k)\\
&= \sum_{k=1}^{m-1} 2^{k}\mathrm{e}(m-1,k) - \sum_{k=1}^{m-1}(k+1)\mathrm{e}(m-1,k)\\
&= \sum_{k=1}^{m-1} 2^{k}\mathrm{e}(m-1,k) - C_m\\
&= \frac{1}{2}\sum_{k=1}^{m-1} 2^{k+1}\mathrm{e}(m-1,k) - \frac{1}{2}C_{m-1} + \frac{1}{2}C_{m-1} - C_m \qquad (6.4)\\
&= \frac{1}{2}s(n-1) + \frac{1}{2}C_{m-1} - C_m\\
&= \frac{1}{2}\binom{n-1}{\lfloor (n-1)/2 \rfloor} + \frac{1}{2}C_{m-1} - C_m\\
&= \frac{1}{2}\binom{2m-1}{m-1} + \frac{1}{2}C_{m-1} - C_m
\end{aligned}$$

reflexive noncrossing partitions of $[n]$ constructed in this way.

Therefore, adding up the numbers in Equations 6.1, 6.2, 6.3 and 6.4, which represent the total number of reflexive noncrossing partitions of $[n]$ constructed in the four different ways, we get a total of

$$\binom{2m-1}{m-1}+\frac{1}{2}\binom{2m-1}{m-1}-\frac{1}{2}C_{m-1}+C_m+\frac{1}{2}\binom{2m-1}{m-1}+\frac{1}{2}C_{m-1}-C_m=2\binom{2m-1}{m-1}$$

reflexive noncrossing partitions of $[n]$ when $n=2m>2$. But

$$2\binom{2m-1}{m-1}=2\frac{(2m-1)!}{(m-1)!m!}=\frac{(2m)!}{m!m!}=\binom{2m}{m}=\binom{n}{\lfloor n/2\rfloor}.$$

□

Proof of Corollary 6.1. $s(n)=\binom{n}{\lfloor n/2\rfloor}$ counts the number of orbits of size one, since it counts the number of noncrossing partitions of $[n]$ fixed by the action of s. Those noncrossing partitions not fixed by the action of s are paired into orbits of size two. There are

$$|\mathrm{NC}_n|-\binom{n}{\lfloor n/2\rfloor}=C_n-\binom{n}{\lfloor n/2\rfloor}$$

noncrossing partitions of $[n]$ not fixed by s, hence there are

$$\frac{1}{2}\left[C_n-\binom{n}{\lfloor n/2\rfloor}\right]$$

orbits of size two. □

The sequence $s(1), s(2), s(3), \ldots$ is A001405 in [16], shifted one term to the right. The first terms of this sequence are

$$1, 2, 3, 6, 10, 20, 35, 70, 126, 252, \ldots$$

Notice that these are the entries of the last column of Figure 25 beginning with row $n=1$.

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