Global asymptotic stability of solutions of nonautonomous master equations

Berton A. Earnshaw and James P. Keener
Department of Mathematics, University of Utah, Salt Lake City, Utah

Introduction

Let $X : \mathbb{R}_+ \to \{x_1, \ldots, x_n\}$ be a jump process such that the *transition probabilities*

$$p(x_i, t|x_j, s) = \text{Prob}\{X(t) = x_i \mid X(s) = x_j\} \quad (t \geq s \geq 0)$$

satisfy the *Chapman-Kolmogorov equations* (CKEs)

$$p(x_i, t|x_j, s) = \sum_{k=1}^{n} p(x_i, t|x_k, u)p(x_k, u|x_j, s) \quad (t \geq u \geq s).$$

Assuming that the transition probabilities are of the form

$$p(x_i, t + \Delta t|x_j, t) = a_{ij}(t)\Delta t + o(\Delta t) \quad (t \geq 0),$$

where $a_{ij}(t) \geq 0$ for all $t \geq 0$, the CKEs gives rise to a system of linear ordinary differential equations for the probability distribution $\mathbf{p}(t)$ of $X(t)$ called the *master equation*,

$$\frac{d\mathbf{p}}{dt} = A(t)\mathbf{p}. \tag{1}$$

Here $A(t)$ is an $n \times n$ matrix whose off-diagonal entries are the *transition rates* $a_{ij}(t)$ and whose columns sum to zero. These conditions ensure that the sum of the entries of a solution of (1) is conserved and that nonnegative solutions remain nonnegative. Such matrices are called $\mathbb{W}$-matrices by van Kampen[1].

van Kampen's theorem for autonomous master eqs

A $\mathbb{W}$-matrix M is called *decomposable* if there exists a permutation matrix P such that

$$P^{-1}MP = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix},$$

and is called *splitting* if

$$P^{-1}MP = \begin{bmatrix} M_1 & 0 & N_1 \\ 0 & M_2 & N_2 \\ 0 & 0 & M_3 \end{bmatrix}.$$

van Kampen's Theorem. *Suppose A is a constant $\mathbb{W}$-matrix. If A is neither decomposable nor splitting, then every probability distribution solution of the master equation (1) approaches a unique stationary distribution.*

The proof of this theorem follows from three facts about $\mathbb{W}$-matrices.

1. All $\mathbb{W}$-matrices have zero as an eigenvalue, due to the column sum condition.
2. If λ is a nonzero eigenvalue of a $\mathbb{W}$-matrix then $\Re(\lambda)$ is negative, due to the column sum condition and the fact that all off-diagonal entries are nonnegative.
3. The only $\mathbb{W}$-matrices with a null space of dimension two or more are those which are decomposable or splitting.

Thus for $\mathbb{W}$-matrices A which are neither decomposable nor splitting, there is a unique vector in the one-dimensional null space of A which is a probability distribution, and it is globally attracting in the affine hyperplane of all vectors whose entries sum to one.

References

1. van Kampen. *Stochastic processes in physics and chemistry*. Elsevier, Amsterdam, 1992.
2. Earnshaw & Keener. *Global asymptotic stability of solutions of nonautonomous master equations*. Submitted.
3. Earnshaw & Keener. *On the existence of invariant manifolds of binomial-like nonautonomous master equations*. Submitted.

Acknowledgements

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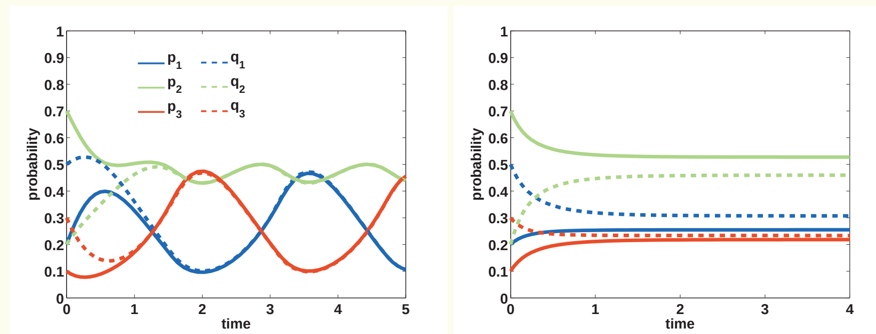
A theorem for nonautonomous master equations?

We wish to generalize van Kampen's theorem to include nonautonomous master equations. Such a theorem is important in applications, e.g. in ion channel kinetics, where subunit opening and closing rates are subject to external forces (e.g. membrane voltage) and are therefore often unavoidably nonautonomous.

There are two issues to consider when moving to the nonautonomous case. The first is that, in general, nonautonomous master equations may have no stationary distributions. For example, take

$$A(t) = \begin{bmatrix} -2\alpha(t) & \beta(t) & 0 \\ 2\alpha(t) & -\alpha(t) - \beta(t) & 2\beta(t) \\ 0 & \alpha(t) & -2\beta(t) \end{bmatrix}$$

with $\alpha(t) = |\sin(t)|$ and $\beta(t) = |\cos(t)|$. Note that this transition rate matrix is neither decomposable nor splitting for all t, however the corresponding master equation has no stationary distributions. Nevertheless, all probability distribution solutions do approach each other in time, as illustrated in the simulations on the left.



The second issue is that even if the transition rate matrix is neither decomposable nor splitting for all t, solutions still may not approach each other in time due to the asymptotic behavior of A, and this is true even if the master equation possesses a stationary distribution. For example, take the transition rate matrix above with $\alpha(t) = \beta(t) = \exp(-2t)$. The resulting matrix $A(t)$ is neither decomposable nor splitting for all t, and the corresponding master equation possesses a stationary distribution, yet these solutions do not converge to each other.

A conjecture for nonautonomous master equations

In light of these issues, we say that a probability distribution solution $\mathbf{p}$ of the master equation (1) is *globally asymptotically stable* (GAS) if for every other probability distribution solution $\mathbf{q}$,

$$\lim_{t\to\infty} (\mathbf{p}(t) - \mathbf{q}(t)) = \mathbf{0}.$$

Thus van Kampen's theorem states that when A is neither decomposable nor splitting, the probability distribution solutions of (1) are GAS. We believe that the correct generalization of this theorem is the following[2]:

Conjecture. *Let $A : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ be a continuous, $\mathbb{W}$-matrix-valued function, and let $\lambda_1(t), \ldots, \lambda_n(t)$ be an ordering of the n eigenvalues of $A(t)$, counting multiplicities, such that $0 = \lambda_1(t) \geq \Re(\lambda_2(t)) \geq \cdots \geq \Re(\lambda_n(t))$ for all $t \geq 0$. If $\Re(\lambda_2)$ is not integrable on the half-line $\mathbb{R}_+$, then the probability distribution solutions of the master equation (1) are GAS.*

We currently have no proof for this conjecture. However, in what follows we present special cases of this conjecture which we have proven, each one providing different conditions on the transition rate matrix A under which the probability distribution solutions of (1) are GAS. We also demonstrate that the converse of this conjecture is not true in general by providing a counterexample.

Knowing when the probability distribution solutions of the master equation (1) are GAS is useful when (1) also possesses a low-dimensional invariant manifold, since all solutions will necessarily be attracted to it. If one can restrict the master equation to the invariant manifold, one then obtains a simpler description of the asymptotic behavior of all probability distribution solutions. We have recently shown that a large class of nonautonomous master equations have one-dimensional invariant manifolds that are loosely related to the manifold of binomial distributions[3].

Special cases of the conjecture

Our first theorem is a straightforward extension of van Kampen's theorem.

Theorem. *Suppose $A(t) = f(t)M$ for all $t \geq 0$, where M is constant $\mathbb{W}$-matrix and $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous. The probability distribution solutions of the master equation (1) are GAS if and only if M is neither decomposable nor splitting and f is not integrable.*

The proof is similar to the proof of van Kampen's theorem, since in this case the fundamental matrix solution of (1) is still $\exp\left(\int_0^t A(s)ds\right) = \exp\left[\left(\int_0^t f(s)ds\right) M\right]$. This is a special case of the conjecture since the eigenvalues of $A(t)$ are the eigenvalues of M scaled by $f(t)$.

The next theorem addresses transition rate matrices A which are asymptotically periodic (including asymptotically constant). We call the set of accumulation points of A its *ω-limit set*, $\omega(A) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} A(t)}$.

Theorem. *If A is continuous and $\mathbb{W}$-matrix-valued and there exists a continuous, periodic, $\mathbb{W}$-matrix-valued function B whose ω-limit set contains at least one matrix that is neither decomposable nor splitting such that*

$$\lim_{t\to\infty} ||A(t) - B(t)||_1 = 0,$$

then the probability distribution solutions of the master equation (1) are GAS.

The proof follows from the fact that when t is large, the $\mathcal{L}^1$-norm of the difference of any two probability distribution solutions must decrease by some uniform amount during each period of B. This is a special case of the conjecture since λ_2 asymptotically approaches a continuous, nonpositive, periodic function which is negative at least once during each period.

Our last theorem addresses less-structed transition rate matrices.

Theorem. *If A is a differentiable, $\mathbb{W}$-matrix-valued function such that both A and its derivative are bounded, and the ω-limit set of A contains no matrix which is either decomposable or splitting, then the probability distribution solutions of the master equation (1) are GAS.*

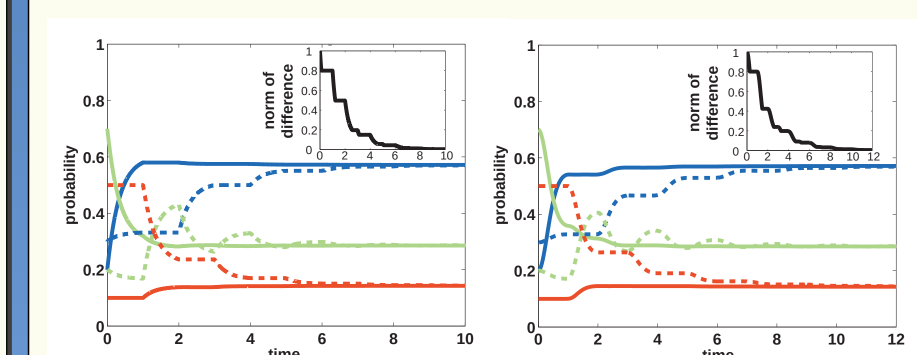
In the proof of this theorem we show that if $||\mathbf{p}(t) - \mathbf{q}(t)||_1 \to r > 0$, then $\omega(A)$ contains a decomposable or splitting matrix. This a special case of the conjecture since $\omega(\lambda_2)$ is nonempty and contains a negative number and $\lambda_2'(t)$ is bounded.

The converse of the conjecture is not true in general

We now present a transition rate matrix A which has $\lambda_2 \equiv 0$, yet whose probability distribution solutions are GAS. Take

$$A(t) = \begin{cases} A_1, & t \in [0, 1), \\ A_2, & t \in [1, 2), \end{cases} \quad A_1 = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

The proof is straightforward. Simulation results are shown on the left. On the right are shown simulation results when taking $|\sin(t)|\, A(t)$, which is continuous.



Therefore, our conjecture does not completely determine all master equations (1) whose probability distribution solutions are GAS.